§ IV.1. The Transport Equation

We consider the one-dimensional radiative transfer equation for a leaf canopy confined between depths \( z = 0 \) at the top and \( z = z_H \) at the bottom, that is the vertical ordinate is directed downwards. All directions are measured with respect to \(-z\) such that \( \mu > 0 \) for upward travelling directions. The canopy is assumed bounded at the bottom by a reflecting and absorbing ground and illuminated at the top by a monodirectional beam source (direct solar radiation) of intensity \( I_o \) along \( \Omega \), and a diffuse source (skylight) of intensity \( I_d \). The appropriate transfer equation is,

\[
-\mu \frac{\partial}{\partial z} I(z, \Omega) + \sigma(z, \Omega) I(z, \Omega) = \int_{4\pi} d\Omega' \sigma_s(z, \Omega' \rightarrow \Omega) I(z, \Omega'), \tag{4.1}
\]

with the boundary conditions,

\[
I(z = 0, \Omega) = I_o \delta(\Omega - \Omega_o) + I_d(\Omega), \quad \mu < 0, \tag{4.2a}
\]

\[
I(z = z_H, \Omega) = \frac{1}{\pi} \int_{2\pi-} d\Omega' \rho_s(\Omega' \rightarrow \Omega) |\mu'| I(z = z_H, \Omega'), \quad \mu > 0, \mu' < 0. \tag{4.2b}
\]

In the above, \( \sigma \) is the total interaction cross section, \( \sigma_s \) is the differential scattering cross section and \( \rho_s \) is the soil bidirectional reflectance distribution function. It is convenient to express the incident field as,

\[
I_o = \frac{f_{\text{dir}}}{\pi \mu_o} F_z(z = 0),
\]

\[
I_d = (1 - f_{\text{dir}}) d_0(z = 0, \Omega) F_z(z = 0),
\]

where \( f_{\text{dir}} \) is the fraction of total incident flux density at the top of the canopy, \( F_z(z = 0) \), in the beam form and \( d_0 \) is the anisotropy of the diffuse source [see Eq. (3.24)]. The incident radiation field is assumed monochromatic, that is, confined to a narrow wavelength interval.

Assume that the leaf normal orientation distribution function (§ II.1) and the leaf scattering phase function (§ II.2) are independent of depth in the canopy, in which case

\[
\sigma(z, \Omega) = u_L(z) G(\Omega), \tag{4.3a}
\]

\[
\sigma_s(z, \Omega' \rightarrow \Omega) = u_L(z) \frac{1}{\pi} \Gamma(\Omega' \rightarrow \Omega), \tag{4.3b}
\]

where \( u_L \) is the leaf area density distribution [Eq. (2.1)], \( G \) is the geometry factor [Eq. (2.14)] and \( \Gamma \) is the area scattering phase function [Eq. (2.17)] introduced previously. Dividing Eq. (4.1) through by \( u_L \) we can change the vertical coordinate from depth \( z \) to cumulative leaf area index \( L \), such that the canopy is now contained between \( L = 0 \) at the top and \( L = L_H \) at the bottom (\( L_H \) is the leaf area index of the canopy).

§ IV.2. Separation of Uncollided and Collided Intensities

It is convenient for numerical purposes and also to gain insight on the transport physics to separate the uncollided radiation field from the collided field. Let

\[
I(L, \Omega) = P_o(L, \Omega) + P(L, \Omega), \tag{4.4}
\]
where $P$ is the specific intensity of uncollided photons and $F$ is the specific intensity of photons which experienced collisions with the elements of the host medium. By introducing Eq. (4.4) in Eq. (4.1), the transport equation can be split into equations for the uncollided

$$-\mu \frac{\partial}{\partial L} P(L, \Omega) + G(\Omega) P(L, \Omega) = 0 \ , \quad (4.5a)$$

$$F(L = 0, \Omega) = I_0 \delta(\Omega - \Omega_0) + L_d(\Omega), \quad \mu < 0, \quad (4.5b)$$

$$F(L = L_H, \Omega) = \frac{1}{\pi} \int_{2\pi} d\Omega' \rho_s(\Omega' \rightarrow \Omega') |\mu'| F'(L = L_H, \Omega'), \mu > 0, \mu' < 0 \ , \quad (4.5c)$$

and collided

$$-\mu \frac{\partial}{\partial L} F(L, \Omega) + G(\Omega) F(L, \Omega) = Q(L, \Omega) + S(L, \Omega), \quad (4.6a)$$

$$F(L = 0, \Omega) = 0, \quad \mu < 0, \quad (4.6b)$$

$$F(L = L_H, \Omega) = \frac{1}{\pi} \int_{2\pi} d\Omega' \rho_s(\Omega' \rightarrow \Omega') |\mu'| F'(L = L_H, \Omega'), \mu > 0, \mu' < 0 \ , \quad (4.6c)$$

problems. In the above, $Q$ is the first collision source

$$Q(L, \Omega) = \frac{1}{\pi} \int_{4\pi} d\Omega' \Gamma(\Omega' \rightarrow \Omega) F'(L, \Omega'), \quad (4.7)$$

and $S$ is the distributed source

$$S(L, \Omega) = \frac{1}{\pi} \int_{4\pi} d\Omega' \Gamma(\Omega' \rightarrow \Omega) F'(L, \Omega'). \quad (4.8)$$

§IV.3. The Uncollided Problem

The solution to the uncollided problem [Eqs. (4.5)] is

$$P^0(L, \Omega) = P^0(L = 0, \Omega) P[\Omega_2, (L - 0)], \quad \mu < 0, \quad (4.9a)$$

$$P^0(L, \Omega) = P^0(L = L_H, \Omega) P[\Omega_2, (L_H - L)], \quad \mu > 0. \quad (4.9b)$$

where

$$P[\Omega_2, (L_2 - L_1)] = \exp \left[- \frac{1}{|\mu|} G(\Omega) (L_2 - L_1) \right], \quad (4.10)$$

denotes the probability of photons not experiencing collisions while travelling along $\Omega$ between depths $L_1$ and $L_2 (L_2 > L_1)$. The downward uncollided intensity at the top of the canopy $P^0(L = 0, \Omega)$ in Eq. (4.9a) is given by the boundary condition Eq. (4.5a). The upward intensity at the ground $P^0(L = L_H, \Omega)$ in Eq. (4.9b) can be evaluated as (cf. Eq. 4.5b),

$$F^0(L = L_H, \Omega) = \frac{1}{\pi} \int_{2\pi} d\Omega' \rho_s(\Omega' \rightarrow \Omega') |\mu'| F'(L = L_H, \Omega'), \quad \mu > 0, \mu' < 0 .$$
§IV.4. The First Collision Problem

The collided problem specified by Eq. (4.6) is, of course, difficult to solve because of the distributed source term. Analytical solutions are possible only in the case of simple scattering kernels. As noted previously (§ II.4), $\Gamma$ is, in general, not rotationally-invariant, and this precludes the use of many standard techniques developed in transport theory. If scattering in the medium is weak, a single-scattering approximation may suffice. The corresponding transport problem,

$$\begin{align}
-\mu \frac{\partial}{\partial L} F(L, \Omega) + G(\Omega) F(L, \Omega) &= Q(L, \Omega), \\
F(L = 0, \Omega) &= 0, \quad \mu < 0, \\
F(L = L_H, \Omega) &= \frac{1}{\pi} \int_{2\pi -} d\Omega' \rho_s(\Omega' \rightarrow \Omega) |\mu'| F(L = L_H, \Omega'), \mu > 0, \mu' < 0 \quad (4.11c)
\end{align}$$

can be solved for the single-scattered intensity, that is, radiation intensity of photons scattered once, $I^1$, with just the first collision source term,

$$\begin{align}
I^1(L, \Omega) &= \frac{1}{|\mu|} \int_0^L dL' Q(L', \Omega) P[\Omega, (L - L')] \quad (4.12a) \\
I^1(L, \Omega) &= \frac{1}{|\mu|} \int_0^{L_H} dL' Q(L', \Omega) P[\Omega, (L' - L)] \quad (4.12b)
\end{align}$$

§IV.5. Successive Orders of Scattering Approximation

The specific intensity of photons which experienced two collisions in the medium $I^2$ can be solved with knowledge of first scattered intensity $I^1$ as follows,

$$\begin{align}
I^2(L, \Omega) &= \frac{1}{|\mu|} \int_0^L dL' S_1(L', \Omega) P[\Omega, (L - L')] \quad (4.13a) \\
I^2(L, \Omega) &= \frac{1}{|\mu|} \int_0^{L_H} dL' S_1(L', \Omega) P[\Omega, (L' - L)] \quad (4.13b)
\end{align}$$

where the distributed source term evaluated with first scattered intensity $S_1$ is

$$S_1(L, \Omega) = \frac{1}{\pi} \int_{4\pi} d\Omega' \Gamma(\Omega' \rightarrow \Omega) I^1(L, \Omega'). \quad (4.14)$$

The foregoing may be generalized for $n$-th order of scattering as,

$$\begin{align}
I^n(L, \Omega) &= \frac{1}{|\mu|} \int_0^L dL' S_{n-1}(L', \Omega) P[\Omega, (L - L')] \quad (4.15a) \\
I^n(L, \Omega) &= \frac{1}{|\mu|} \int_0^{L_H} dL' S_{n-1}(L', \Omega) P[\Omega, (L' - L)] \quad (4.15b)
\end{align}$$

The total intensity $I$ can be evaluated as simply

$$I(L, \Omega) = I^0(L, \Omega) + \sum_{n=1}^{\infty} I^n(L, \Omega). \quad (4.16)$$

In practise, the summation in Eq. (4.16) is limited to $N$-orders of scattering. The value of $N$ depends on the leaf area index and the single scattering albedo of the leaves [Eq. (2.8)]. The value of this method is its intuitive nature and is not recommended for numerical calculations as other elegant and precise methods are available. This method has been applied to model vegetation reflection by Myneni et al. (1987).
§IV.6. Gauss-Seidel Iteration Procedure for the Collided Problem

Consider the collided problem specified by Eq. (4.6). Discretize the angular variable $\Omega$ into a finite number of directions, $\Omega_j$, $j = 1, 2, ..., M$, with the weights denoted by $w_j$. Similarly, discretize the spatial variable $L_i$ that is, divide $L_H$ into $N$ layers, each of thickness $\Delta L$. We use the notation $L_i$ and $L_{i+1}$ to denote successive layers. The collided radiation at $L_{i+2}$ and $L_i$ in down and upward directions can be written as,

$$
\Gamma(L_{i+2}, \Omega_j) = \frac{1}{|\mu_j|} \int_{L_i}^{L_{i+2}} dL' \ J(L', \Omega_j) \ P[\Omega_j, (L_{i+2} - L')] \ , \quad \mu < 0 \ , \quad (4.17a)
$$

$$
\Gamma(L_i, \Omega_j) = \frac{1}{|\mu_j|} \int_{L_i}^{L_H} dL' \ J(L', \Omega_j) \ P[\Omega_j, (L' - L_i)] \ , \quad \mu > 0 \ , \quad (4.17b)
$$

where $J = Q + S$ is the source term, that is, the sum of first collision and distributed sources. Eq. (4.17a) can be rewritten as

$$
\Gamma(L_{i+2}, \Omega_j) = \Gamma(L_i, \Omega_j) + \frac{1}{|\mu_j|} \int_{L_i}^{L_{i+2}} dL' \ J(L', \Omega_j) \ P[\Omega_j, (L_{i+2} - L')] \ , \quad \mu < 0 \ .
$$

If $\Delta L$ is small, then the following approximation is valid,

$$
\frac{1}{|\mu_j|} \int_{L_i}^{L_{i+2}} dL' \ J(L', \Omega_j) \ P[\Omega_j, (L_{i+2} - L')] = \frac{1}{|\mu_j|} \int_{L_i}^{L_{i+2}} dL' \ J(L', \Omega_j) \ P[\Omega_j, (L' - L_i)]
$$

$$
\equiv J(L_{i+1}, \Omega_j) \left(1 - \exp \left[-\frac{1}{|\mu_j|} G(\Omega_j) \ 2\Delta L \right] \right) \ , \quad (4.18)
$$

where the source in the intervening layer is evaluated as

$$
J(L_{i+1}, \Omega_j) = \frac{1}{\pi} \sum_{k=1}^{M} \ w_k \Gamma(\Omega_k \rightarrow \Omega_j) \ \Gamma(L_{i+1}, \Omega_k) + Q(L_{i+1}, \Omega_j) \ . \quad (4.19)
$$

Eq. (4.17b) can be simplified to similar expressions in an analogous way.

The systems of linear algebraic equations (4.19) can be solved iteratively for the unknowns $\Gamma(L_i, \Omega_j)$, $i = 1, 2, ..., N$, $j = 1, 2, ..., M$, using the Gauss-Seidel iteration procedure. The downward and upward intensities are computed from layer to layer for every iteration step. For instance, in the $n$-th iteration, the values of the downward intensities in layer $i + 2$ are computed from the downward intensities of layer $i$ and $i + 1$ of the same iteration step and from the upward intensities of layer $i + 1$ of the previous iteration step $n - 1$.

The advantages of this method are that the internal radiation field is readily available without additional labor and it is possible to treat vertical inhomogeneities. The main limitation of this method is that it becomes tedious in optical deep canopies at strongly scattering wavelengths. This method has been used to model vegetation reflectance by Knyazikhin and Marshak (1991) and, Liang and Strahler, (1993).

§IV.7. Discrete Ordinates Method for the Collided Problem

We now consider numerical solution of the transport problem for the collided intensity using the discrete ordinates method as developed in neutron transport theory. In this method,
photons are restricted to travel in a finite number of discrete directions, usually the quadrature
directions, such that the angular integrals are evaluated with high precision. The spatial
derivatives may be approximated by a finite difference scheme, to result in an set of equations
which can be used to solve for the collided radiation field by iterating on the distributed
source. This method has been used successfully to model the radiation regime in vegetation
canopies by Shultis and Myneni (1988).

We consider the transport problem for the collided intensity [Eq. (4.6)]. The angular
dependence of the transport equation is approximated by discretizing the angular variables $\mu$
and $\phi$ into a set of $[N \times M]$ discrete directions. The source terms are evaluated by numerical
quadrature where $[\mu_i, \phi_j]$ are the quadrature ordinates and the set of corresponding weights
are $[w_i, w_j]$. The transport equation for the collided intensity [Eq. (4.6)] can be written as

$$-\mu_i \frac{\partial}{\partial L} F(L, \Omega_{ij}) + G(\Omega_{i}) F(L, \Omega_{ij}) = Q(L, \Omega_{ij}) + S(L, \Omega_{ij}), \quad (4.20)$$

where the first collision and distributed sources are

$$Q(L, \Omega_{ij}) = \frac{1}{\pi} \sum_{m=1}^{M} w_m \sum_{n=1}^{N} w_n \Gamma(\Omega_{nm} \rightarrow \Omega_{ij}) F'(L, \Omega_{nm}), \quad (4.21a)$$

$$S(L, \Omega_{ij}) = \frac{1}{\pi} \sum_{m=1}^{M} w_m \sum_{n=1}^{N} w_n \Gamma(\Omega_{nm} \rightarrow \Omega_{ij}) F'(L, \Omega_{nm}). \quad (4.21b)$$

The vegetation canopy contained between $L = 0$ and $L = L_H$ is divided into $K$ layers of equal
thickness $\Delta L$. The spatial derivative in Eq. (4.20) is approximated as

$$\frac{\partial}{\partial L} F'(L_{k+0.5}, \Omega_{ij}) = \frac{[F'(L_{k+1}, \Omega_{ij}) - F'(L_k, \Omega_{ij})]}{\Delta L} \quad (4.22)$$

where $k + 0.5$ is the center of the layer between the edges $k$ and $k + 1$. The discretized version
of transport equation thus reads as

$$-\mu_i \frac{[F'(L_{k+1}, \Omega_{ij}) - F'(L_k, \Omega_{ij})]}{\Delta L} + G(\Omega_{i}) F'(L_{k+0.5}, \Omega_{ij}) = Q(L_{k+0.5}, \Omega_{ij}) + S(L_{k+0.5}, \Omega_{ij}), \quad (4.23)$$

with $k = 1, 2, ..., K, i = 1, 2, ..., N$ and $j = 1, 2, ..., M$. To reduce the number of unknowns,
a relation between cell-edge and cell-center collided intensities is required. Typically the
following is used,

$$F'(L_{k+0.5}, \Omega_{ij}) \approx (1 - \alpha) F'(L_k, \Omega_{ij}) + \alpha F'(L_{k+1}, \Omega_{ij}), \quad \mu < 0, \quad (4.24a)$$

$$F'(L_{k+0.5}, \Omega_{ij}) \approx (1 - \alpha) F'(L_{k+1}, \Omega_{ij}) + \alpha F'(L_k, \Omega_{ij}), \quad \mu > 0, \quad (4.24b)$$

and if $\alpha = 0.5$, the cell-center intensity is the arithmetic average of the cell-edge intensities.

Equation (4.23) can be solved for $F'(L_{k+1}, \Omega_{ij})$ in terms of $F'(L_k, \Omega_{ij})$ in view of Eqs.
(4.24) as

$$F'(L_{k+1}, \Omega_{ij}) = a_{ij} F'(L_k, \Omega_{ij}) - b_{ij} J(L_{k+0.5}, \Omega_{ij}), \quad \mu < 0, \quad (4.25)$$

and for $F'(L_k, \Omega_{ij})$ in terms of $F'(L_{k+1}, \Omega_{ij})$ as

$$F'(L_k, \Omega_{ij}) = c_{ij} F'(L_{k+1}, \Omega_{ij}) + d_{ij} J(L_{k+0.5}, \Omega_{ij}), \quad \mu > 0. \quad (4.26)$$
In the above,

\[
\begin{align*}
    a_{ij} &= \frac{[1 + f_{ij}(1 - \alpha)]}{[1 - f_{ij}\alpha]}, \\
    b_{ij} &= \frac{f_{ij}}{[1 - f_{ij}\alpha]}, \\
    c_{ij} &= \frac{[1 - f_{ij}(1 - \alpha)]}{[1 + f_{ij}\alpha]}, \\
    d_{ij} &= \frac{f_{ij}}{[1 + f_{ij}\alpha]}, \\
    f_{ij} &= \frac{1}{\mu_i} G(\Omega_{ij}) \Delta L, \\
    J(L_{k+0.5}, \Omega_{ij}) &= Q(L_{k+0.5}, \Omega_{ij}) + S(L_{k+0.5}, \Omega_{ij}).
\end{align*}
\]

These equations are of the standard form except for the angular dependence of the coefficients \(a_{ij}\) through \(d_{ij}\) because of the geometry factor \(G\). The set of Eqs. (4.25) and (4.26) can be used to solve for collided intensity as follows. While sweeping downwards in the phase space, Eq. (4.25) is used to step successively down in the mesh. At the bottom, the boundary condition is handled as

\[
F^d(L_{K+1}, \Omega_{ij}) = \frac{1}{\pi} \sum_{m=1}^{M} \sum_{n=1}^{N/2} w_n \rho, (\Omega_{um} \rightarrow \Omega_{ij}) |\mu_n| F^d(L_{K+1}, \Omega_{um}).
\]

Note that \(i = (N/2) + 1, \ldots, N\) in the above. Now, Eq. (4.26) is used to sweep through the grid in upward directions. The distributed source is upgraded [Eq. (4.21b)] using the relations between cell-edge and cell-center intensities [Eqs. (4.24)]. This procedure is repeated until the cell-edge intensities in successive iterations do not differ by more than a preset threshold value.

\section{IV.8. Two-Stream Approximations}

In cases where the angular distribution of the radiation field is of less interest, the transport equation can be angle-integrated to derive the appropriate equations for radiation fluxes. One example is the evaluation of hemispherical reflectances BHR or DHR. The resulting differential equations can be solved analytically in some cases. Because of its simplicity and the possibility of analytical solutions, methods based on flux approximations have been widely used to model vegetation canopy radiation regime (Allen and Richardson, 1968; Suits, 1972; Verhoef, 1984; Sellers, 1985; amongst others).

We consider the transport problem stated by Eq. (4.1). We first consider the downward flux density \(F^d\) defined as

\[
F^d(L) = \int_{-\pi}^{\pi} d\Omega \ |\mu| I(L, \Omega).
\]

Integrating Eq. (4.1) over all downward directions, but with change of vertical coordinate \(z\) to cumulative leaf area index \(L\),

\[
\frac{\partial}{\partial L} F^d(L) + F^d(L) \int_{-\pi}^{\pi} d\Omega G(\Omega) I(L, \Omega) = F^u(L) \int_{-\pi}^{\pi} d\Omega \int_{-\pi}^{\pi} d\Omega' \frac{1}{\pi} G(\Omega' \rightarrow \Omega) I(L, \Omega') + F^d(L) \int_{-\pi}^{\pi} d\Omega \int_{-\pi}^{\pi} d\Omega' \frac{1}{\pi} G(\Omega' \rightarrow \Omega) I(L, \Omega'),
\]

(4.29)
and simplifying results in a differential equation for the downward flux density,

$$\frac{\partial}{\partial L} F^d(L) + K_1^d F^d(L) = K_2^d F^u(L) + K_3^d F^d(L). \tag{4.30}$$

Similarly, a differential equation for the upwards flux density can be derived,

$$-\frac{\partial}{\partial L} F^u(L) + K_1^u F^u(L) = K_2^u F^u(L) + K_3^u F^d(L). \tag{4.31}$$

The initial values for Eqs. (4.30) and (4.31) are,

$$F^d(L = 0) = f_{\text{dir}} |\mu_o| L_o + (1 - f_{\text{dir}}) \int_{2\pi} d\Omega_i |\mu_d| I_d(\Omega_d), \tag{4.32}$$

$$F^u(L = L_H) = r_s F^d(L = L_H), \tag{4.33}$$

where $r_s$ is the hemispherical reflectance of the ground underneath the canopy. While these initial value problems seem simple enough, it is not easy to rigorously derive expressions for the coefficients $K$ in the general case of distributed leaf normals and anisotropic scattering kernels. However, approximate expressions generally suffice in many practical instances.

We consider the simple case of a horizontally homogeneous leaf canopy consisting of horizontal leaves. The geometry factor $G(\Omega) = |\mu|$ and the area scattering phase function is simply

$$\Gamma(\Omega' \rightarrow \Omega) = \begin{cases} \tau_L |\mu|, & \mu > 0, \\ \rho_L |\mu|, & \mu < 0. \end{cases} \tag{4.34a}$$

on the assumption of negligible specular reflection from leaf surfaces. In the above $\rho_L$ and $\tau_L$ are the leaf hemispherical reflectance and transmittance. The coefficients for this canopy are, $K_1^d = 1, K_2^d = \rho_L, K_3^d = \tau_L, K_1^u = 1, K_2^u = \tau_L, \text{ and } K_3^u = \rho_L$. The differential equations (4.30) and (4.31) can be rewritten as

$$\frac{\partial}{\partial L} F^d(L) = \rho_L F^u(L) + (\tau_L - 1) F^d(L), \tag{4.34a}$$

$$-\frac{\partial}{\partial L} F^u(L) = \rho_L F^d(L) + (\tau_L - 1) F^u(L). \tag{4.34b}$$

That is, the changes in the downward flux density are given by the sum of backscattered upward flux density (the gain term) and the fraction of downward flux density that is not forward scattered (the loss term). Similarly, the changes in the upward flux density are given by the sum of backscattered downward flux density (the gain term) and the fraction of upward flux density that is not forward scattered (the loss term). The transport equations can be solved with the initial values [Eqs. (4.32) and (4.33)] to obtain downward and upward radiation flux density in the medium.

If the leaves are uniformly distributed, the geometry factor $G(\Omega) = 0.5$ and the area scattering phase function is given by

$$\Gamma(\Omega' \rightarrow \Omega) = \frac{\omega_L}{4\pi} (\sin \beta - \beta \cos \beta) + \frac{\tau_L}{\pi} \cos \beta, \tag{4.35}$$

where $\beta = \arccos(\Omega' \cdot \Omega)$, again on the assumption of negligible specular reflection from leaf surfaces. Analytical expressions for the coefficients $K$ and solution of the corresponding transport equations for downward and upwards fluxes are straightforward, although tedious.
§IV.9. The Hot-Spot Effect

The hot spot effect results from considerations of the relative sizes of the scatterers in the canopy (leaves, branches, twigs, etc.) and the wavelength of the radiation. Shadowing is not only ubiquitous, but the far-field assumption is rarely satisfied in reality. Consequently, mutual shadowing of scatterers is predominant. The reflected radiation field tends to peak about the retro-illumination direction under such cases, and this is termed the hot spot effect in vegetation remote sensing. The shape and magnitude of the hot spot depends on the structure of the medium and is especially pronounced at shorter wavelengths where scattering is weak (shadows are darker). The hot spot phenomenon is observational evidence of the limitations of theoretical developments that ignore scatterer size and resulting directional correlations in the interaction cross sections. Inclusion of such considerations in the transport equation is feasible, but complicated (Myneni et al., 1991). Here we present a simple methodology for inclusion of the hot spot phenomenon in the transport equation. An elegant formulation of the hot spot effect in the limit of single scattering can be found in Kuusk (1985).

We include the hot spot effect in the transport equation through the use of a modified total interaction cross section $\tilde{\sigma}$ (cf. Marshak, 1989),

$$
\tilde{\sigma}(L, \Omega, \Omega') = \begin{cases} 
\sigma(L, \Omega) \{1 - \exp \left( -\kappa D(\Omega, \Omega') \right) \}, & (\Omega \cdot \Omega') < 0, \\
\sigma(L, \Omega), & (\Omega \cdot \Omega') > 0,
\end{cases}
$$

(4.35)

where $\kappa$ is a parameter related to the ratio of vegetation height to characteristic leaf dimension. Its values was estimated to be between 1 and 8 from experimental data. The distance $D$ is given by

$$
D(\Omega, \Omega') = \left[ \frac{1}{\mu^2} + \frac{1}{\mu'^2} + \frac{2(\Omega \cdot \Omega')}{\mu \mu'} \right]^{0.5}.
$$

This particular model for the modified total interaction cross section has two desirable features, namely, that for $\Omega = -\Omega'$, $\tilde{\sigma}$ vanishes to result in the hot spot, and for large scattering angles, it approaches the standard cross section $\sigma$. Note that $\tilde{\sigma}$ is also always positive.

We consider the one-dimensional leaf canopy transport problem. Let the total radiation intensity be represented as $I = I^0 + I^1 + I^m$, that is, the sum of uncollided, first-collision and multiple-collision intensities. Further, for ease of presentation, let us assume that the incident diffuse skylight $I_d = 0$, that is, $f_{dir} = 1$. The transport problems for $I^0$ and $I^1$ specified by equations (4.5) and (4.11) can be modified by using $\tilde{\sigma}$ instead of $\sigma$ or equivalently $\tilde{G}$ instead of $G$ [cf. Eqs. (4.3)]. The solutions for the downward $I^0$ and $I^1$ given by Eqs. (4.9a) and (4.12a) remain unchanged since $\tilde{G} = G$ for $(\Omega \cdot \Omega') > 0$. The upward uncollided and first-collided radiation intensities are, however, modified because of the modified cross section. They read,

$$
I^0(L, \Omega) = I^0(L = L_H, \Omega) P[\Omega, \Omega,(L_H - L)] , \quad \mu > 0.
$$

(4.36a)

$$
I^1(L, \Omega) = \frac{1}{|\mu|} \int_L^{L_H} dL' Q(L', \Omega) P[\Omega, \Omega,(L' - L)] , \quad \mu > 0.
$$

(4.36b)

where

$$
P[\Omega, \Omega,(L_2 - L_1)] = \exp \left[ -\frac{1}{|\mu|} \tilde{G}(\Omega, \Omega_3) (L_2 - L_1) \right],
$$

(4.37)

denotes the probability of photons not experiencing collisions while travelling along $\Omega_3$ from the top of the canopy ($L = 0$) to depth $L_2$ and along $\Omega$ from $L_2$ and $L_1$ ($L_2 > L_1$). This is the required bi-directional gap probability for implementing the hot spot effect.
The multiple-collision transport problem is similar to the collided intensity transport problem specified by Eqs. (4.6) except that the first collision source \( Q \) in Eq. (4.6a) is replaced by the \textit{second-collision source}

\[
Q(L, \Omega) = \frac{1}{\pi} \int_{4\pi} d\Omega' \Gamma(\Omega' \rightarrow \Omega) I^1(L, \Omega') ,
\]

The above formulation allows simulation of the hot spot effect with transport equations. This is accomplished in an \textit{ad hoc} manner by utilizing a modified interaction cross section for the uncollided and first-collided intensities arising from incident solar radiation, and then using the standard cross section for multiply-collided transport problem. If needed, the uncollided and collided intensities due to incident diffuse skylight can be solved the standard way utilizing the unmixed cross section.

While the above formalism allows inclusion of the hot spot effect, it does result in a system that \textit{violates the energy conservation principle}, since the transport problem for the first-collision intensity is, strictly speaking,

\[
-\mu \frac{\partial}{\partial L} I^1(L, \Omega) + G(\Omega) I^1(L, \Omega) = \frac{1}{\pi} \int_{4\pi} d\Omega' \Gamma(\Omega' \rightarrow \Omega) I^0(L, \Omega') + [G(\Omega) - \tilde{G}(\Omega, \Omega)] I^1(L, \Omega) ,
\]

\[
I^1(L = 0, \Omega) = 0 , \quad \mu < 0 ,
\]

\[
I^1(L = L_H, \Omega) = \frac{1}{\pi} \int_{2\pi} d\Omega' \rho_s(\Omega' \rightarrow \Omega) |\mu'| I^1(L = L_H, \Omega') , \quad \mu > 0 , \quad \mu' < 0 .
\]

This is equivalent to a transport equation for the total intensity \( I \) [Eqs. (4.1)] with an additional \textit{fictitious} internal source, \([G(\Omega) - \tilde{G}(\Omega, \Omega)] I^1(L, \Omega)\), which results in violation of the energy conservation principle. This has serious implications for the inverse problems, where this principle is used as a constraint.

\textbf{§IV.10. The Discrete Ordinates Method in Three Spatial Dimensions}

We consider a leaf canopy contained between \( 0 < z < Z_S, 0 < x < X_S, 0 < y < Y_S \), where \( X_S, Y_S \) and \( Z_S \) denote the dimensions of the stand. The canopy is assumed illuminated only on the top surface. At the bottom of the canopy, the ground is assumed to both reflect and absorb the radiation field. The governing transport equation is

\[
-\mu \frac{\partial}{\partial z} I(\mathbf{r}, \Omega) + \eta \frac{\partial}{\partial y} I(\mathbf{r}, \Omega) + \xi \frac{\partial}{\partial x} I(\mathbf{r}, \Omega) + \sigma(\mathbf{r}, \Omega) I(\mathbf{r}, \Omega) = J(\mathbf{r}, \Omega) ,
\]

where \( \mathbf{r} \equiv (x, y, z) \), \( \mu, \eta \) and \( \xi \) are the directional cosines with respect to \( x, y \) and \( z \) coordinates, and the source term \( J \) is

\[
J(\mathbf{r}, \Omega) = \int_{4\pi} d\Omega' \sigma_s(\mathbf{r}, \Omega' \rightarrow \Omega) I(\mathbf{r}, \Omega') .
\]

The corresponding boundary conditions are,

\[
I(x, y, z = 0, \Omega) = I_o \delta(\Omega - \Omega_o) + I_d(\Omega) , \quad \mu < 0 ,
\]

\[
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\]
\[
I(x, y, z, \Omega) = \frac{1}{\pi} \int_{2\pi} d\Omega \rho_s(\Omega^* \rightarrow \Omega) \mu' I(x, y, z, \Omega'), \quad \mu > 0, \quad \mu' < 0, \quad (4.40b)
\]
\[
I(x = 0, y, z, \Omega) = 0, \quad \xi > 0, \quad (4.40c)
\]
\[
I(x = X_S, y, z, \Omega) = 0, \quad \xi < 0, \quad (4.40d)
\]
\[
I(x, y = 0, z, \Omega) = 0, \quad \eta > 0, \quad (4.40e)
\]
\[
I(x, y = Y_S, z, \Omega) = 0, \quad \eta < 0. \quad (4.40f)
\]

In some instances, it may be desireable to simulate a canopy of larger horizontal extent with the following

\[
I(x = 0, y, z, \Omega) = I(x = X_S, y, z, \Omega), \quad \xi > 0, \quad (4.41a)
\]
\[
I(x = X_S, y, z, \Omega) = I(x = 0, y, z, \Omega), \quad \xi < 0, \quad (4.41b)
\]
\[
I(x, y = 0, z, \Omega) = I(x, y = Y_S, z, \Omega), \quad \eta > 0, \quad (4.41c)
\]
\[
I(x, y = Y_S, z, \Omega) = I(x, y = 0, z, \Omega), \quad \eta < 0. \quad (4.41d)
\]

We introduce a fine spatial mesh by dividing the domain into cells bounded by \(x_{1/2}, x_{3/2}, \ldots, x_{K+1/2}\) (of width \(\Delta x\)), \(y_{1/2}, y_{3/2}, \ldots, y_{J+1/2}\) (of width \(\Delta y\)), and \(z_{1/2}, z_{3/2}, \ldots, z_{I+1/2}\) (of width \(\Delta z\)). The cross sections \(\sigma\) and \(\sigma_s\) are assumed to be piece-wise constant and can take new values only at the cell-boundaries. Within the cell volume \(V_{ijk}\), bounded by \((x_{k-1/2} < x < x_{k+1/2}), (y_{j-1/2} < y < y_{j+1/2})\) and \((z_{i-1/2} < z < z_{i+1/2})\), the cross sections are denoted as \(\sigma_{ijk}\) and \(\sigma_{s,ijk}\).

Introducing the first-order finite-difference approximation [Eq. (4.22)] for the spatial derivatives in the angle-discretized 3D transport equation [Eq. (4.38)], and integrating over the cell volume, we obtain

\[
-\mu_i \int_j dy \int_k dx \left[ I_n(x, y, z_{i+1/2}) - I_n(x, y, z_{i-1/2}) \right] + \\
\eta_j \int_k dz \int_i dx \left[ I_n(x, y_{j+1/2}, z) - I_n(x, y_{j-1/2}, z) \right] + \\
\xi_k \int_i dz \int_j dy \left[ I_n(x_{k+1/2}, y, z) - I_n(x_{k-1/2}, y, z) \right] + \\
\sigma_{n,ijk} \int_i dz \int_j dy \int_k dx \left[ I_n(x, y, z) \right] = \\
= \int_i dz \int_j dy \int_k dx \left[ J_n(x, y, z) \right], \quad (4.42)
\]

where \(\int_k dx\) denotes integration from \(x_{k-1/2}\) to \(x_{k+1/2}\), and so on. The subscript \(n\) denotes the discrete direction of photon travel. Dividing Eq. (4.42) by the cell volume \(V_{ijk} = \Delta x \Delta y \Delta z\) results in

\[
\frac{-\mu_i}{\Delta z} \left[ I_{n,ijk}(z_{i+1/2}) - I_{n,ijk}(z_{i-1/2}) \right] + \frac{\eta_j}{\Delta y} \left[ I_{n,ik}(y_{j+1/2}) - I_{n,ik}(y_{j-1/2}) \right] + \\
\frac{\xi_k}{\Delta x} \left[ I_{n,ij}(x_{k+1/2}) - I_{n,ij}(x_{k-1/2}) \right] + \sigma_{n,ijk} I_{n,ijk} = J_{n,ijk}, \quad (4.43)
\]

In the above, the average radiation intensities over the cell surfaces are,

\[
I_{n,ijk}(z_{i\pm 1/2}) = \frac{1}{\Delta y \Delta z} \int_j dy \int_k dx \left[ I_n(x, y, z_{i\pm 1/2}) \right], \quad (4.44a)
\]

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\[ I_{n, i k}(y_{j + 1/2}) = \frac{1}{\Delta z \Delta x} \int_i dz \int_k dx \; I_n(x, y_{i + 1/2}, z), \quad (4.44b) \]
\[ I_{n, ij}(x_{k + 1/2}) = \frac{1}{\Delta z \Delta y} \int_i dz \int_j dy \; I_n(x_{i + 1/2}, y, z), \quad (4.44c) \]

Similarly, the averages over the cell volume of the specific intensity and the total source are,

\[ I_{nij} = \frac{1}{V_{ijk}} \int_i dz \int_j dy \int_k dx \; I_n(x, y, z) , \quad (4.45a) \]
\[ J_{nij} = \frac{1}{V_{ijk}} \int_i dz \int_j dy \int_k dx \; J_n(x, y, z) , \quad (4.45b) \]

Equation (4.3) is exact but not closed. To solve for the cell-center angular intensities \( I_{nij} \) and the flows across the three surfaces through which photons can leave the cell volume, three additional relations are required (note that the flows across the three surfaces through which photons enter the cell are known either from the boundary conditions or from previous calculations). The difference relations, introduced earlier [Eq. (4.24)], can be used for this purpose,

\[ I_{nij} \approx 0.5[I_{nijk}(z_{i+1/2}) + I_{nijk}(z_{i-1/2})] , \quad (4.46a) \]
\[ I_{nij} \approx 0.5[I_{nijk}(y_{j+1/2}) + I_{nijk}(y_{j-1/2})] , \quad (4.46b) \]
\[ I_{nij} \approx 0.5[I_{nijk}(x_{k+1/2}) + I_{nijk}(x_{k-1/2})] . \quad (4.46c) \]

These relations are simple but can lead to negative intensities, in which case remedies must be implemented in the algorithm. The simplest thing to do is set the offending intensity to zero and proceed with the calculation.

In this manner, the angular and spatial dependence of the transport equation is discretized while ensuring that the condition of positivity, symmetry and balance are satisfied. In each octant, the incoming and outgoing flows are identified depending on the sign of the direction cosines in order not to violate the principle of directional evaluation, that is, sweeping in the phase-space along the direction of photon flow only. Using Eqs. (4.46), the exiting flows can be eliminated to solve for the cell center intensity. A generic equation for the cell center intensity can be written as

\[ I_{nijk} = \frac{J_{nijk} + \frac{2\eta_n}{\Delta z} I_{nijk}(z_{i+1/2}) + \frac{2\eta_n}{\Delta y} I_{nijk}(y_{j+1/2}) + \frac{2\eta_n}{\Delta x} I_{nijk}(x_{k+1/2})}{\sigma_{nijk} + \frac{2\rho_n}{\Delta z} + \frac{2\rho_n}{\Delta y} + \frac{2\rho_n}{\Delta x}} . \quad (4.47) \]

The three flows in the numerator represent the incoming flows across the three faces of the cell and are specific to an octant. The cell center intensity evaluated with Eq. (4.47) is then used in the relations given in Eqs. (4.46) to evaluate the three outgoing flows from the cell. For example, in octant 1, \( \mu_n, \eta_n \) and \( \xi_n \) are positive. The three incoming flows are \( I_{nijk}(z_{i+1/2}), I_{nijk}(y_{j-1/2}) \) and \( I_{nijk}(x_{k+1/2}) \). The outgoing flows to be evaluated are \( I_{nijk}(z_{i-1/2}), I_{nijk}(y_{j+1/2}) \) and \( I_{nijk}(x_{k-1/2}) \). This phase-space sweeping along the direction of photon travel is embedded in an iteration on the distributed source with appropriate convergence criteria built in. Details on the implementation and acceleration techniques for the iterative procedure can be found in Myneni et al. (1990).

\[ \textbf{IV.11. References} \]

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