I.1 The Radiation Field

**Photon**: The energy in the radiation field is assumed carried by point massless particles called photons. The energy of a photon \( E \) is \( hv \), where \( h \) is Planck’s constant and \( v \) is photon frequency.

**Particle Distribution Function**: Let \( f(r, \nu, \Omega, t) \) denote the distribution function such that

\[
\text{dn} = f \text{ dr dv dr},
\]

that is, \( \text{dn} \) is the number of photons, at time \( t \), at space point \( r \) in a differential volume element \( \text{dr} \) with frequency \( \nu \) in a frequency interval \( \text{dv} \), and traveling in a direction \( \Omega \) in a solid angle element \( \text{d} \Omega \). In plane geometry,

\[
\text{dr} = \text{dx dy dz}, \quad \text{d} \Omega = \sin \theta \text{ d} \theta \text{ d} \phi = \text{d} \mu \text{ d} \phi,
\]

where \( \mu = \cos \theta \)

**Specific Intensity**: The definition of specific intensity \( I \) is

\[
I(r, \nu, \Omega, t) = \text{chv} f(r, \nu, \Omega, t),
\]

where \( c \) is vacuum speed of light. Its physical interpretation is contained in the relationship (cf. Fig. I.1),

\[
dE = I(r, \nu, \Omega, t) \cos \theta \text{ d} \nu \text{ d} \Omega \text{ d} \sigma \text{ dt}.
\]

Here \( dE \) is the amount of radiant energy in \( \text{dv} \) centered at \( \nu \), traveling in a direction \( \Omega \) confined to a solid angle element \( \text{d} \Omega \), which crosses, in a time element \( \text{dt} \), an area \( \text{d} \sigma \) oriented such that \( \theta \) is the angle which the direction \( \Omega \) makes with the normal to \( \sigma \) (\( \cos \theta = \Omega \cdot \text{n} \)).

If the specific intensity is independent of \( \Omega \) at a point, it is said to be *isotropic* at that point. If the intensity is independent of both \( r \) and \( \Omega \), the radiation field is said to be *homogeneous* and *isotropic*.

**Energy Density**: Energy density \( u \) is the first angular moment of the specific intensity,

\[
u(r, t) = \frac{1}{c} \int_{0}^{\infty} \int_{4\pi} d\Omega I(r, \nu, \Omega, t).
\]

**Radiative Flux**: The rate of energy flow per unit area across a surface is defined as the radiative flux. It is a vector quantity. For instance, the \( x \) component of the radiative flux \( F_x \) is the flow of energy across a unit surface area of an element oriented perpendicular to the \( x \) axis, that is,
\[ F_x(r, t) = \int_0^\infty d\nu \int d\Omega x r(r, \nu, \Omega, t), \quad (1.4) \]

where \( \Omega_x \) is the projection of \( \Omega \) along the \( x \) axis. In a similar way, \( F_y \) and \( F_z \) can be defined.

**Radiation Pressure:** The rate of momentum flow across a surface is defined as pressure. For example, the component \( p_{xy} \) is defined as the rate of \( y \) momentum flow per unit area through a surface perpendicular to the \( x \) axis, and is given by,

\[ p_{xy}(r, t) = \frac{1}{c} \int_0^\infty d\nu \int d\Omega x r(r, \nu, \Omega, t), \quad (1.5) \]

Note: The momentum of a photon is \( h\nu/c \). The other 8 components of the pressure tensor can be similarly defined.

### I.2. Interaction of the Radiation Field with Matter

**Absorption:** The absorption coefficient \( \sigma_a \) is defined such that the probability of a photon being absorbed in traveling a distance \( ds \) is \( \sigma_a (r, \nu, \Omega, t) ds \). The dependence on the direction of photon travel is noteworthy and is especially important in the case of vegetation media.

**Scattering:** The scattering coefficient \( \sigma_s' \) is defined in analogy to the absorption coefficient, that is,

\[ \text{Probability of scattering} = \sigma_s' (r, \nu, \Omega, t) ds. \]

Unlike absorption, a scattering event serves to change the direction and/or frequency of the incident photon. Thus, it is convenient to define a differential scattering coefficient \( \sigma_s \) as,

\[ \text{Probability of scattering} = \sigma_s (r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) ds. \]

The change in photon frequency as a result of a scattering event is not relevant in optical remote sensing of vegetation. It is important to note that photon scattering in vegetation media depends on the absolute coordinates of \( \Omega' \) and \( \Omega \) in general. The scattering coefficients \( \sigma_s' \) and \( \sigma_s \) are related as

\[ \sigma_s'(r, \nu', \Omega', t) = \int_0^\infty d\nu \int d\Omega \cdot \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t). \quad (1.6) \]

In some cases, the differential scattering coefficient is decomposed into the product

\[ \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = \sigma_s'(r, \nu', \Omega', t) \cdot K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t), \quad (1.7) \]
such that, the kernel $K$ has the interpretation of a probability density function,

$$
\int_0^\infty d\nu \int_\Omega d\Omega \ K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = 1.
$$

(1.8)

In the case of coherent scattering, there is no frequency change upon scattering and,

$$
K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = K(r, \Omega' \rightarrow \Omega, t) \delta(\nu' - \nu).
$$

In the case of isotropic scattering,

$$
K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = \frac{1}{4\pi} K(r, \nu' \rightarrow \nu, t).
$$

Therefore, the simplest scattering kernel responds to isotropic coherent scattering,

$$
K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = \frac{1}{4\pi} \delta(\nu' - \nu).
$$

(1.9)

**Extinction**: The extinction, or the total interaction, coefficient $\sigma$ is simply the sum $\sigma_s + \sigma'$. Therefore, $\sigma(r, \nu, \Omega, t) ds$ is the probability that a photon would disappear from the beam while traveling a distance $ds$ in the medium (note that it can reappear at a different frequency and/or direction.) The quantity $1/\sigma$ denotes *photon mean free path*, that is, the average distance a photon will travel in the medium before suffering a collision.

**Single Scattering Albedo**: The probability of scattering given that a collision has occurred is given by the single scattering albedo, $\omega = \sigma_s/\sigma$. In the case of conservative scattering, $\omega = 1$. The case $\omega = 0$ denotes pure absorption.

**Emission**: Photons can be introduced into the medium through the external and/or internal sources. The number of photons emitted per unit time and volume at frequency $\nu$ in interval $d\nu$ and direction in $\Omega$ in $d\Omega$ is $q(r, \nu, \Omega, t) dv d\Omega$.

**I.3. The Equation of Transfer**

With Eq. (1.1) in mind, we consider a “cube” in six dimensional space of dimensions $\Delta x, \Delta y, \Delta v, \Delta \mu, \Delta \phi$. The number of photons in this cube at time $t$ is

$$
f(r, \nu, \Omega, t) \Delta x \Delta y \Delta z \Delta v \Delta \mu \Delta \phi,
$$

where $f$ is the photon distribution function. The time rate of change of the number of photons is
\[ \text{change} = \Delta V \frac{\partial}{\partial t} f(r, \nu, \Omega, t), \quad (1.10) \]

where \( \Delta V = \Delta x \Delta y \Delta z \Delta \nu \Delta \mu \Delta \phi \). This time rate change is due to five processes –

[1] Streaming: The net rate of photons streaming out of the cube along the \( x \) dimension is

\[ \text{streaming} = \Delta V \frac{\partial}{\partial x} [\hat{x} f(r, \nu, \Omega, t)], \]

where \( \hat{x} \) denotes the \( x \) component of the photon velocity \( \hat{\chi} = c\Omega_x \). The total net rate of streaming from the cube along all the six dimensions is

\[ \text{streaming} = \Delta V \left[ \frac{\partial}{\partial x} (\hat{x}f) + \frac{\partial}{\partial y} (\hat{y}f) + \frac{\partial}{\partial z} (\hat{z}f) + \frac{\partial}{\partial \nu} (\hat{\nu}f) + \frac{\partial}{\partial \mu} (\hat{\mu}f) + \frac{\partial}{\partial \phi} (\hat{\phi}f) \right], \quad (1.11) \]

where \( f \equiv f(r, \nu, \Omega, t) \).

[2] Absorption: The rate of absorption in \( \Delta V \) is the product of the number of photons in the cube, \( f\Delta V \), and the probability of absorption per photon and per unit time, \( c\sigma_a \),

\[ \text{absorption} = c \sigma_a (r, \nu, \Omega, t) f(r, \nu, \Omega, t) \Delta V. \quad (1.12) \]

[3] Outscattering: The rate of photon loss due to outscattering from \( \nu, \Omega \) to all other frequencies and directions is the product of the number of photons in the cube and the probability of outscattering per photon and per unit time,

\[ \text{outscattering} = c \Delta V \int_0^\infty \int_0^{4\pi} d\Omega' \sigma_a (r, \nu \rightarrow \nu', \Omega \rightarrow \Omega', t) f(r, \nu, \Omega, t), \]

\[ = c \Delta V \sigma_a (r, \nu, \Omega, t) f(r, \nu, \Omega, t). \quad (1.13) \]

[4] Inscattering: The rate of photon gain due to inscattering to \( \nu, \Omega \) from all other frequencies and directions is the product of the number of photons in the cube and the probability of inscattering per photon and per unit time,

\[ \text{inscattering} = c \Delta V \int_0^\infty \int_0^{4\pi} d\Omega' \sigma_a (r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) f(r, \nu', \Omega', t). \quad (1.14) \]

[5] Emission: The rate of the production of photons in the cube is simply

\[ \text{emission} = q(r, \nu, \Omega, t) \Delta V. \quad (1.15) \]
Transfer Equation: The equation of transfer is essentially a statement of photon number conservation arrived at by equating the sum of the five terms, Eqs. (1.1) to (1.15), with appropriate signs to designate a loss or gain, to the overall rate of change given by Eq. (1.10):

\[
\text{Time rate of change} = - \text{streaming} - \text{absorption} - \text{outscattering} + \text{inscattering} + \text{emission},
\]

or,

\[
\frac{\partial}{\partial t} f(v, \Omega) + \frac{\partial(xf)}{\partial x} + \frac{\partial(yf)}{\partial y} + \frac{\partial(zf)}{\partial z} + \frac{\partial(vf)}{\partial v} + \frac{\partial(\mu f)}{\partial \mu} + \frac{\partial(\hat{\phi} f)}{\partial \hat{\phi}}
+ c \ \sigma_s(v, \Omega) f(v, \Omega) + c \ \sigma'(v, \Omega) f(v, \Omega)
= c \int_0^\infty dv' \int_4 \sigma'(v' \rightarrow v, \Omega' \rightarrow \Omega) f(v', \Omega') q(v, \Omega),
\]

where we dropped all arguments \( r \) and \( t \), as well as \( v \) and \( \Omega \) in the streaming terms, and cancelled the common term \( \Delta V \). Note that –

- \( \hat{\mu} = \hat{\phi} = 0 \) as these angles are measured with respect to fixed axes in space,
- \( \hat{v} = 0 \) as photons stream in straight lines between collisions with no change in frequency,
- \( \hat{x} = c\Omega_x, \hat{y} = c\Omega_y, \hat{z} = c\Omega_z \), as photons stream with speed \( c \) along the direction \( \Omega \);

thus, the equation of transfer can be written as [cf. Eqs. (1.2) and the definition of the extinction coefficient],

\[
\frac{1}{c} \frac{\partial}{\partial t} I(v, \Omega) + \Omega \cdot \nabla I(v, \Omega) + \sigma(v, \Omega) I(v, \Omega)
= S(v, \Omega) + \int_0^\infty dv' \int_4 \sigma'(v' \rightarrow v, \Omega' \rightarrow \Omega) I(v', \Omega'),
\]

where \( S = hvq \), the rate of energy emission. Sometimes, the second and the third terms on the left hand side are grouped together; the term \( [\Omega \cdot \nabla + \sigma] \) then denotes the streaming-collision operator. The transfer equation is clearly an integro-differential equation, which because of its partly differential nature, requires both spatial and temporal boundary conditions.

I.4. Boundary Conditions

Boundary conditions in space and time variables are required since the equation of transfer is a first order differential equation in these variables. We consider a host medium of arbitrary composition and shape, and assume that the medium is non-re-entrant, that is, photons exiting the medium do not re-enter the medium. The spatial boundary condition specifies the radiation field incident at all points on the surface,

\[
I(r_s, v, \Omega, t) = \Gamma(r_s, v, \Omega, t), \quad n \cdot \Omega < 0,
\]

(1.18)
where \( \Gamma \) is a specified function, \( r_s \) is a point on the surface of the medium and \( \mathbf{n} \) is an outward normal vector at this point. The case of vacuum boundary condition refers to \( \Gamma = 0 \). The temporal boundary condition is

\[
I(r, \nu, \Omega, 0) = \Lambda(r, \nu, \Omega),
\]

on the assumption that the temporal range of interest is \( 0 < t < \infty \). The radiative transfer problem is thus completely specified by the equation of transfer [Eq. (1.17)] and the two boundary conditions [Eqs. (1.18) and (1.19)].

1.5. The Equation of Transfer in Plane Geometry

The streaming term in the equation of transfer \( \mathbf{\Omega} \cdot \nabla I \), is the directional derivative of the specific intensity in the \( \mathbf{\Omega} \) direction, that is,

\[
\mathbf{\Omega} \cdot \nabla I = \frac{dI}{ds},
\]

\[
= \frac{\partial I}{\partial x} \left( \frac{dx}{ds} \right) + \frac{\partial I}{\partial y} \left( \frac{dy}{ds} \right) + \frac{\partial I}{\partial z} \left( \frac{dz}{ds} \right),
\]

\[
= \frac{\partial I}{\partial x} \mathbf{\Omega} \cdot \mathbf{e}_x + \frac{\partial I}{\partial y} \mathbf{\Omega} \cdot \mathbf{e}_y + \frac{\partial I}{\partial z} \mathbf{\Omega} \cdot \mathbf{e}_z
\]

where \( s \) is a length along \( \mathbf{\Omega} \). In plane geometry, the explicit expression for the streaming term is,

\[
\mathbf{\Omega} \cdot \nabla I = \frac{\partial I}{\partial x} (\sin \theta \cos \phi) + \frac{\partial I}{\partial y} (\sin \theta \sin \phi) + \frac{\partial I}{\partial z} (\cos \theta),
\]

\[
= \xi \frac{\partial I}{\partial x} + \eta \frac{\partial I}{\partial y} + \nu \frac{\partial I}{\partial z}.
\]

Radiative transfer problems in vegetation remote sensing studies may be considered time independent and generally involve no frequency shifting interactions. The relevant equation of transfer in plane geometry is

\[
-\mu \frac{\partial I}{\partial z} + \eta \frac{\partial I}{\partial y} + \xi \frac{\partial I}{\partial x} + \sigma(r, \Omega) I(r, \Omega) =
\]

\[
S(r, \Omega) + \int d\Omega' \sigma(r, \Omega' \rightarrow \Omega) I(r, \Omega'),
\]

\[
(1.21)
\]
where \( z \) axis denotes depth and the polar angle \( \theta \) is measured with respect to \(-z\); hence the negative sign. The emission term \( S \) is required at thermal wavelengths, but \( S=0 \) at solar wavelengths.

### I.6. The Equation of Transfer in Integral Form

It is instructive to derive an integral form of the equation of transfer for the physical insight it provides into the process of radiation transport. We rewrite Eq. (1.21) as

\[
\Omega \cdot \nabla I(r, \Omega) + \sigma(r, \Omega) I(r, \Omega) = Q(r, \Omega),
\]

(1.22)

where \( Q \) denotes the angular source due to inscattering [assume that \( S=0 \) in the right hand side of Eq. (1.21)]. The corresponding spatial boundary condition is

\[
I(r_s, \Omega) = \Gamma(r_s, \Omega), \quad n \cdot \Omega < 0.
\]

(1.23)

Let the length \( s \) be the distance back along the direction \( \Omega \) from the point \( r \). Equation (1.22) can be written as

\[
- \frac{dI(r-s\Omega, \Omega)}{ds} + \sigma(r-s\Omega, \Omega) I(r-s\Omega, \Omega) = Q(r-s\Omega, \Omega).
\]

Multiplying the above with exponential integrating factors,

\[
\exp \left[ - \int_{S_0}^{S} \sigma(r-s''\Omega, \Omega) \right],
\]

yields,

\[
- \frac{d}{ds} \left\{ I(r-s\Omega, \Omega) \exp \left[ - \int_{S_0}^{S} \sigma(r-s''\Omega, \Omega) \right] \right\}
\]

\[
= Q(r-s\Omega, \Omega) \exp \left[ - \int_{S_0}^{S} \sigma(r-s''\Omega, \Omega) \right],
\]

where \( s_0 \) is an arbitrary point along \( s \). Integration of the above from \( s \) to \( s_0 \) and manipulation yields

\[
I(r-s\Omega, \Omega) = I(r-s_0\Omega, \Omega) \exp \left[ \int_{S_0}^{S} \sigma(r-s''\Omega, \Omega) \right] +
\]

\[
\int_{S}^{S_0} ds' Q(r-s'\Omega, \Omega) \exp \left[ \int_{S_0}^{S} \sigma(r-s''\Omega, \Omega) \right].
\]
Setting $s=0$ in the above results in an equation for $I(\vec{r}, \Omega)$,

$$I(\vec{r}, \Omega) = I(\vec{r} - s, \Omega, \Omega) \exp \left[ -\int_{s_0}^{s} \sigma(r - s'', \Omega, \Omega) \, ds'' \right] +$$

$$\int_{s_0}^{s} d's Q(r - s', \Omega, \Omega) \exp \left[ -\int_{0}^{s'} \sigma(r - s'', \Omega, \Omega) \, ds'' \right].$$

The boundary condition [Eq. (1.23)] can be used to evaluate $I(r - s_0, \Omega, \Omega)$ by choosing $s_0$ such that $r - s_0 \Omega = r_0$, a point on the exterior boundary of the host medium. Recognizing $s_0 = |r - r_0|$ and making use of Eq. (1.23), we can rewrite the above as

$$I(\vec{r}, \Omega) = \Gamma(\vec{r}, \Omega, \Omega) \exp \left[ -\int_{0}^{\vec{r} - r_0} ds'' \sigma(r - s'', \Omega, \Omega) \right] +$$

$$\int_{0}^{\vec{r} - r_0} ds' Q(r - s', \Omega, \Omega) \exp \left[ -\int_{0}^{s'} \sigma(r - s'', \Omega, \Omega) \, ds'' \right]. \quad (1.24)$$

Equation (1.24) is the desired integral equation for the specific intensity $I$; it can be recognized as a “formal” solution to the radiative transfer equation [Eq. (1.22)] subject to the boundary condition given by Eq. (1.23). Note that this equation needs to be solved for the collided intensity $I$ since the inscattering part of the source term depends on $I$. However, in the special case of no scattering, Eq. (1.24) is the general solution to the transfer equation [Eq. (1.22)].

The physical interpretation of Eq. (1.24) is as follows (Fig. 1.2). The specific intensity $I$ at a point $r$ in the direction $\Omega$ is the sum of two terms – uncollided and collided intensities. The uncollided part is the boundary intensity $I(\vec{r}, \Omega)$ exponentially attenuated by collisions over the distance between the boundary point $r_0$ and the point $r$. The collided part is the contribution to the specific intensity due to inscattering from each path length element $ds'$ along $\Omega$. The radiation energy in each element $ds'$ must be exponentially attenuated to find its contribution to $I(\vec{r}, \Omega)$.

The quantity $\tau(r_0, r, \Omega)$, defined as

$$\tau(r_0, r, \Omega) \equiv \int_{0}^{\vec{r} - r_0} ds'' \sigma(r - s'', \Omega, \Omega), \quad (1.25)$$

is referred to as the optical depth between the points $r$ and $r_0$. The estimation of $\tau$ from measurements of $I$ is a principle problem in remote sensing.
I.7. The Validity of the Equation of Transfer

To what extent is the above description of radiative energy transport valid in reality? The answer requires us to explicitly identify the assumptions made in the derivation of the transport equation (1.17).

[1] *Photons may be treated as point particles*. Photons exhibit wave-like behavior and to treat the wave packet as a point particle, the spread of the packet should be small in both physical and momentum \((v, \Omega)\) space.

[2] *Photon-photon interactions may be neglected*. This means that the photon density is low, that is, low enough such that the overlap in the tails of wavepackets of two photons is negligibly small. This is especially required in the case of source photons emitted at the same location.

[3] *Collisions and emission processes occur instantaneously*. This imposes a limit on the time resolution over which the transport equation is applicable.

[4] *The transport equation does not describe behavior resulting from interference of waves*. Therefore, the transfer equation is valid only when the distance between scatterers is large compared to the wave packets.

[5] *Photons travel in straight lines between collisions*. This requires that the refractive index of the medium be constant both in space and time. If the refractive index changes with position in the medium, a photon will undergo continuous refraction as it streams between collisions. If the refractive index is time dependent, a photon will continuously change its frequency as it streams between collisions.

[6] *The derived equation of transfer assumes unpolarized light*. Four parameters are required to specify the state of polarization of a beam of light, and accordingly, a proper description of photon transport including polarization effects involves four coupled equations of transfer. Assuming the light to be unpolarized by the medium, these equations can be averaged to derive a single equation of transfer and this involves some error.

[7] *Only the expected or mean value of the photon density distribution is considered*. Fluctuations about the mean are not considered, and in some cases, these derivations may be of interest.
Appendix

**Derivation 1**: Derive Transfer Equation in Integral Form

We start the derivation of the integral RTE recalling the differential form,

\[
\vec{\Omega} \cdot \nabla I(\vec{r}, \vec{\Omega}) + \sigma(\vec{r}, \vec{\Omega}) I(\vec{r}, \vec{\Omega}) = Q(\vec{r}, \vec{\Omega}),
\]

with boundary condition

\[
I(\vec{r}_s, \vec{\Omega}) = \Gamma(\vec{r}_s, \vec{\Omega}), \quad \vec{n} \cdot \vec{\Omega} < 0.
\]

Let \( s \) denote the distance back along direction \( \vec{\Omega} \) from location \( \vec{r} \). Using parameter \( s \), equation (1) can be rewritten in the equivalent form

\[
\frac{d}{ds} I(\vec{r} + s\vec{\Omega}, \vec{\Omega}) + \sigma(\vec{r} + s\vec{\Omega}, \vec{\Omega}) I(\vec{r} + s\vec{\Omega}, \vec{\Omega}) = Q(\vec{r} + s\vec{\Omega}, \vec{\Omega}) \quad (3a)
\]

or

\[
-\frac{d}{ds} I(\vec{r} - s\vec{\Omega}, \vec{\Omega}) + \sigma(\vec{r} - s\vec{\Omega}, \vec{\Omega}) I(\vec{r} - s\vec{\Omega}, \vec{\Omega}) = Q(\vec{r} - s\vec{\Omega}, \vec{\Omega}). \quad (3b)
\]

In order to integrate Eq. (3b) first multiply it by an exponential integrating factor, which depends on the interval \([s, s_0]\) of parameter \( s \)

\[
\exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right],
\]

resulting in

\[
-\frac{dI(\vec{r} - s\vec{\Omega}, \vec{\Omega})}{ds} \exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right] + \sigma(\vec{r} - s\vec{\Omega}, \vec{\Omega}) I(\vec{r} - s\vec{\Omega}, \vec{\Omega}) \exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right] =
\]

\[
Q(\vec{r} - s\vec{\Omega}, \vec{\Omega}) \exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right]. \quad (4)
\]

Recall that

\[
\frac{d}{ds}\left(\exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right]\right) = -\sigma(\vec{r} - s\vec{\Omega}, \vec{\Omega}) \exp\left[-\int_{s_0}^{s} \sigma(\vec{r} - s''\vec{\Omega}, \vec{\Omega}) ds''\right]
\]

where we use the formula for differentiation of integral over the upper limit,

\[
\frac{d}{dx}\left(\int_{a}^{x} f(t) dt\right) = f(x).
\]
Now, recalling formula for differentiation of product of two functions,

\[
\frac{d}{dx}[u(x)v(x)] = \frac{d}{dx}[u(x)]v(x) + u(x)\frac{d}{dx}[v(x)] = \frac{d}{dx}[u(x)v(x)]
\]

we rewrite left-hand side of Eq. (4) as follows

\[
- \frac{d}{ds}(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] - I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \frac{d}{ds}\left\{ \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] \right\} = \\
- \frac{d}{ds}\left\{ I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] \right\}.
\]

Using this result we rewrite equation (4) as follows

\[
- \frac{d}{ds}\left\{ I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] \right\} = Q(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right]. \tag{5}
\]

Now let us integrate Eq. (5) over \([s, s_0]\). The left-hand side will integrate as follows:

\[
- \int_{s}^{s_0} ds' \frac{d}{ds'}\left\{ I(\tilde{r} - s'\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s'} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] \right\} = -I(\tilde{r} - s_0\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] + \\
I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right].
\]

And Eq. (5) is now

\[
- I(\tilde{r} - s_0\tilde{\omega}, \tilde{\omega}) \exp(0) + I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] = \\
\int_{s}^{s_0} ds' Q(\tilde{r} - s'\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{s_0}^{s'} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right].
\]

Let \(s_0=0\). This results in

\[
I(\tilde{r}, \tilde{\omega}) = I(\tilde{r} - s\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{0}^{s} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right] - \\
\int_{s}^{s_0} ds' Q(\tilde{r} - s'\tilde{\omega}, \tilde{\omega}) \exp\left[ - \int_{0}^{s'} ds'' \sigma(\tilde{r} - s''\tilde{\omega}, \tilde{\omega}) \right].
\]

Now let \(s=s_0\),
\[ I(\bar{r}, \bar{\Omega}) = I(\bar{r} - s0\bar{\Omega}, \bar{\Omega}) \cdot \exp \left[ - \int_{0}^{s0} ds'' \sigma(\bar{r} - s''\bar{\Omega}, \bar{\Omega}) \right] + \]
\[ \int_{0}^{s0} ds' Q(\bar{r} - s'\bar{\Omega}, \bar{\Omega}) \cdot \exp \left[ - \int_{0}^{s'} ds'' \sigma(\bar{r} - s''\bar{\Omega}, \bar{\Omega}) \right]. \]

Finally, choose \( s0 \) such that \( r - s0 \cdot \bar{\Omega} = \bar{r}_s \) is on boundary and

\[ s0 = |\bar{r} - \bar{r}_s|. \]

So, we can rewrite equation as follows [we use boundary condition (2) here]

\[ I(\bar{r}, \bar{\Omega}) = \Gamma(\bar{r}_s, \bar{\Omega}) \cdot \exp \left[ - \int_{r-r_s}^{r-r_s} ds'' \sigma(\bar{r} - s''\bar{\Omega}, \bar{\Omega}) \right] + \]
\[ \int_{r-r_s}^{r-r_s} ds' Q(\bar{r} - s'\bar{\Omega}, \bar{\Omega}) \cdot \exp \left[ - \int_{0}^{s'} ds'' \sigma(\bar{r} - s''\bar{\Omega}, \bar{\Omega}) \right], \]

which is the integral equation of transport.
Derivation 2: Derive Energy Conservation Law

Consider stationary (time-independent) transport equation,

\[
\frac{dI(r_0 + \xi'\Omega, \Omega)}{d\xi'} + \sigma(r_0 + \xi'\Omega, \Omega)I(r_0 + \xi'\Omega, \Omega) = \int_{\sigma} (r_0 + \xi'\Omega', \Omega' \rightarrow \Omega)I(r_0 + \xi'\Omega', \Omega')d\Omega'
\]

with boundary conditions

\[
I(r_{\text{surface}}, \Omega) = \Gamma(r_{\text{surface}}, \Omega), \quad \Omega \cdot n < 0
\]

Let us integrate this equation over phase-space \((R^3, 4\pi)\). Refer to the following schematic plot during derivations.

A) The first item on left-hand side can be integrated as follows,

\[
\int_{\Omega} d\Omega \int_{V} d^3R \frac{dI(r_0 + \xi'\Omega, \Omega)}{d\xi'} = \int_{\Omega} d\Omega \int_{0}^{\xi_{\text{surf}}} d\xi' \frac{dI(r_0 + \xi'\Omega, \Omega)}{d\xi'} =
\]

\[
\int_{\Omega} d\Omega \int_{S} dS[ I(r_0 + \xi_{\text{surf}}\Omega, \Omega) - I(r_0, \Omega) ] = [dS = n(S) \cdot \Omega \cdot d|S|] =
\]

\[
\int_{\Omega} d\Omega \int_{S} dS |n(r_0 + \xi_{\text{surf}}\Omega) \cdot \Omega| \parallel I(r_0 + \xi_{\text{surf}}\Omega, \Omega) - \int_{\Omega} d\Omega \int_{S} dS |n(r_0) \cdot \Omega| I(r_0, \Omega) =
\]

\[
[r_{\text{surf}} \equiv r_0 + \xi_{\text{surf}}\Omega = r_{\text{surf}}, r_{\text{surf}} \equiv r_0 ] =
\]

\[
\int_{S} d|S| \parallel I(r_{\text{surf}}, \Omega) \cdot \Omega| I(r_{\text{surf}}, \Omega) - \int_{S} d|S| \parallel I(r_{\text{surf}}) \cdot \Omega| I(r_{\text{surf}}, \Omega) = E_{\uparrow} - E_{\downarrow}
\]
B) The second item on left-hand side can be integrated as follows,

\[
\int \frac{d\Omega}{4\pi} \int d^3R \, \sigma(r_0 + \xi', \Omega, \Omega) I(r_0 + \xi', \Omega, \Omega) = \\
\int \frac{d\Omega}{4\pi} \int d^3R \, \sigma_a(r_0 + \xi', \Omega, \Omega, r_I, r_R) I(r_0 + \xi', \Omega, \Omega) = \\
E_a + E_{\text{scat}}
\]

C) Finally the integration of the right-hand side term gives

\[
\int \frac{d\Omega}{4\pi} \int d\Omega' \int d^3R \, \sigma_s(r_0 + \xi', \Omega', \Omega, r_I, r_R) I(r_0 + \xi', \Omega', \Omega') = \\
\left[ \int \frac{d\Omega}{4\pi} \sigma_s(r_0 + \xi', \Omega') = \sigma_{\text{scat}}(r_0 + \xi', \Omega') \right] = \\
\int \frac{d\Omega}{4\pi} \int d^3R \, \sigma_{\text{scat}}(r_0 + \xi', \Omega, \Omega) I(r_0 + \xi', \Omega, \Omega) = E_{\text{scat}}
\]

Combining A, B, and C we have:

\[
E^\uparrow - E^\downarrow + E_a + E_{\text{scat}} = E_{\text{scat}} \Rightarrow \\
E^\downarrow = E^\uparrow + E_a,
\]

that is, input energy is spent on emittance and absorption.