Small-Scale Drop-Size Variability: Empirical Models for Drop-Size-Dependent Clustering in Clouds

ALEXANDER MARSHAK

Climate and Radiation Branch, NASA Goddard Space Flight Center, Greenbelt, Maryland

YURI KNYAZIKHIN

Department of Geography, Boston University, Boston, Massachusetts

MICHAEL L. LARSEN

Department of Physics, Michigan Technological University, Houghton, Michigan

WARREN J. WISCOMBE

Climate and Radiation Branch, NASA Goddard Space Flight Center, Greenbelt, Maryland

(Manuscript received 10 February 2004, in final form 12 July 2004)

ABSTRACT

By analyzing aircraft measurements of individual drop sizes in clouds, it has been shown in a companion paper that the probability of finding a drop of radius r at a linear scale l decreases as $l^{D(r)}$, where $0 \leq D(r) \leq 1$. This paper shows striking examples of the spatial distribution of large cloud drops using models that simulate the observed power laws. In contrast to currently used models that assume homogeneity and a Poisson distribution of cloud drops, these models illustrate strong drop clustering, especially with larger drops. The degree of clustering is determined by the observed exponents $D(r)$. The strong clustering of large drops arises naturally from the observed power-law statistics. This clustering has vital consequences for rain physics, including how fast rain can form. For radiative transfer theory, clustering of large drops enhances their impact on the cloud optical path. The clustering phenomenon also helps explain why remotely sensed cloud drop size is generally larger than that measured in situ.

1. Introduction

Though it is widely assumed that cloud drops are distributed uniformly in space and fluctuations of the number of drops in a given small volume follow Poisson statistics (e.g., Young 1993), there is strong evidence of cloud-drop clustering on a wide range of scales down to centimeter scales (e.g., Hobbs and Rangno 1985; Baker 1992; Pinsky and Khain 2001, 2003; Kostinski and Jameson 1997; Jameson et al. 1998; Davis et al. 1999; Shaw et al. 2002). Clustering can be identified as significant fluctuations in cloud-drop concentration (Jameson et al. 1998), defined as the expectation of the number of drops per volume when volume tends to 0 (Pawlowska and Brenguier 1997). Analyzing Forward Scattering Spectrometer Probe (FSSP) data, Baker (1992) reported a deviation from a Poisson distribution that is characterized by a perfectly random spatial distribution. Pinsky and Khain (2001) studied the fine structure of cloud-drop concentrations using Fast FSSP (Brenguier et al. 1998) measurements, which showed that the degree of drop-concentration fluctuations strongly depends on the drop size. Later Pinsky and Khain (2003) found that drop clusters on centimeter scales are induced by droplet inertia within turbulent flow. Thus, small-scale drop variability carries information about the fine structure of clouds. Davis et al. (1999) assumed scale invariance in cloud liquid water and used fractal characteristics to describe its spatial variability on scales from centimeters to hundreds of meters, while Jameson et al. (1998) and Kostinski and Jameson (2000) studied fluctuations of the number of 50-μm diameter cloud drops per liter using pair-correlation functions. Recently Shaw et al. (2002) argued that the pair-correlation function is the most natural and physically meaningful measure of correlations.

Corresponding author address: Dr. Alexander Marshak, Climate and Radiation Branch, NASA Goddard Space Flight Center, Code 913, Greenbelt, MD 20771.
E-mail: Alexander.Marshak@nasa.gov

Liu and Hallett (1998) and Liu et al. (2002) looked at the problem from another angle: they pointed out that cloud-drop-size distribution is not a scale-independent function but strongly depends on the spatial scale over which the drops are sampled. However, Liu et al. (2002) suggested that observed drop-size distributions are scale independent above a certain saturation scale. For scales smaller than the saturation scale, the drop-size distribution is "ill-defined" and changes substantially from scale to scale. They hypothesized that the unique property of scale dependence requires a new theoretical framework that treats the scale as an independent variable, just as the variables of space and time are treated in the current framework. Such a parameterization, for example, may result in a better representation of clouds in climate models than complicated models with detailed microphysics because of the large range of scales involved (Liu et al. 2002).

In the companion paper, Knyazikhin et al. (2005, hereafter KMLWMM) show that, for sufficiently small volumes, the mean number of drops with a given radius varies proportionally to a drop-size-dependent nonunit power of the volume. The coefficient of proportionality—a generalized drop concentration—and the power are used to parameterize variability of cloud drops at small scales. Using this parameterization, they estimate the direct impact of the small-scale spatial variability of drops on radiative transfer, concluding that current radiative transfer theory underestimates the effect of large drops on cloud optical path.

The present paper complements KMLWMM's results by demonstrating that the clustering of drops is primarily responsible for the observed power law. More specifically, it addresses the following three questions: (i) Why does the observed power law indicate drop clustering? (ii) How can the clustering phenomenon be modeled so as to agree with observations? And, finally, (iii) How do the observed exponents characterize the degree of clustering?

Understanding the spatial distribution and small-scale fluctuations (inhomogeneity) of large drops in clouds is essential to both the cloud physics and atmospheric radiation communities. For cloud physics, it relates to the coalescence growth of raindrops (Twomey 1976); while for radiation it has a strong indirect influence on the radiative properties of clouds through a rapid modification of the cloud-drop-size distribution, and directly through changing optical path length (KMLWMM).

2. Small-scale cloud-drop-size variability

The analyses of the FSSP data acquired during the First International Satellite Cloud Climatology Project (ISCCP) Regional Experiment (FIRE) in July 1987 indicates that the total number $N(r,l)$ of samples at a linear scale l containing drops of radius r follows a power law with a drop-size-dependent exponent $D(r)$ (KMLWMM):

$$N(r, l) \propto l^{-D(r)}. \quad (1)$$

The exponent $D(r)$ is a nonincreasing function of the drop size r and varies between 1 (for small drops) and 0 (for very large drops). If $D(r) = 1$, drops densely fill the space they occupy and the number of "nonempty" samples at a linear scale l is inversely proportional to l and the total number of drops is proportional to l^3. The case $D(r) = 0$ corresponds to a few sparsely distributed individual drops. For $0 < D(r) < 1$, the frequency of drop occurrence decreases with the drop size r; in other words, the probability of finding a drop of radius r at a linear scale l is proportional to $l^{D(r)}$.

To illustrate Eq. (1), Fig. 1 shows variation in $N(r,l)$ for $r = 7 \pm 2$ μm and $r = 23 \pm 2$ μm derived from data acquired during a 2-h flight on 3 March 2000 in Kansas and Oklahoma as part of the Atmospheric Radiation Measurement (ARM) Cloud Intensive Operational Period (IOP; Dong et al. 2002). It is clearly seen that for a scale range of almost three orders of magnitude (from 80 m to 50 km), the total number of samples with drops follows a power law with exponents $D \approx 1$ and $D \approx 1/2$ for small ($r = 7$ μm) and large ($r = 23$ μm) drops, respectively. Figure 2 shows variation in the concentra-

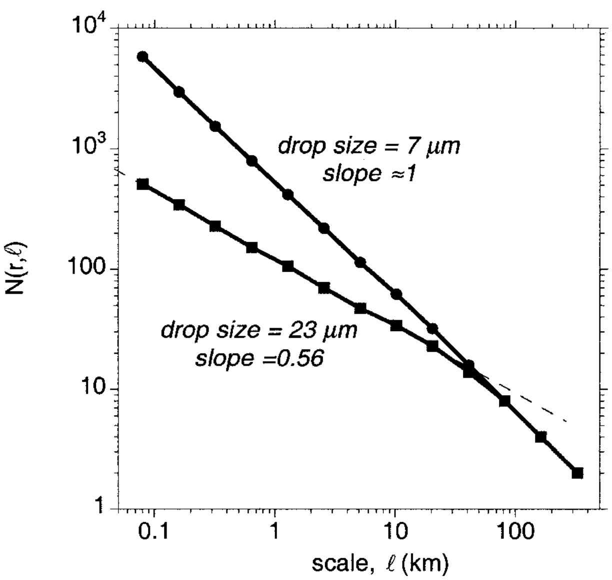

FIG. 1. Number $N(r,l)$, of FSSP samples containing drops with radii $r = 7 \pm 2$ μm (bin 4) and $r = 23 \pm 2$ μm (bin 12) vs scale l derived from data collected by an FSSP on board the University of North Dakota Citation aircraft during the ARM Cloud IOP (Mar 2000). (Available online at a password-protected public site http://iop.archive.arm.gov/arm-iop/2000/sgp/cloud/poellot-citation/.) While for small drops, $D \approx 1$, for large drops at scales between 80 m and 40 km the variation in $N(r,l)$ clearly follows a power law with an exponent $D = 0.56$.

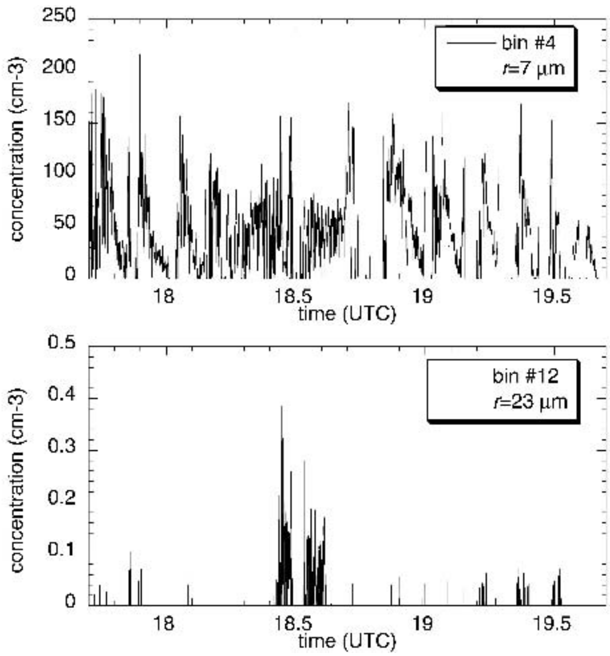

FIG. 2. Concentration of drops with radius (top) $r = 7 \pm 2$ μm (bin 4) and (bottom) $r = 23 \pm 2$ μm (bin 12) for the 2 h of the same flight (3 Mar 2001) as in Fig. 1. Note that small droplets (top panel) almost uniformly fill the space; the set of points on the horizontal axis with positive concentration has a fractal dimension close to 1. In contrast, large drops (bottom panel) are clustered; a fractal dimension of the set with positive concentration is 0.56.

tion of small and large drops in 80-m intervals along the flight path. One can see that while small drops are more likely to be unclustered, large drops are positively clustered, (i.e., detecting a drop makes it more likely that the next drop will be detected nearby). This suggests that the deviation of the exponent from unity indicates a clustering in drop spatial distribution.

What is the importance of Eq. (1) deduced from the analysis of FSSP drop-size distributions? It follows from this equation that

- in contrast to the underlying assumption of radiative transfer theory, the mean number of drops is proportional to the drop-size-dependent power of the volume (Wiscombe et al. 2003);
- such behavior cannot be described by a density distribution function used in data analysis—a cumulative distribution function should be used instead (Knyazikhin et al. 2002);
- for sufficiently small volumes, the mean number of rarer large drops in a given volume decreases more slowly than conventional approaches assume; consequently their radiative impact is underestimated (KMLWMM).

In the next section we show how one can simulate the spatial distribution of drops that follow a power law [(Eq. (1)] with a given exponent. The case $D(r) = 1$ for small drops is well documented in the literature—the drop distribution can be simulated by a Poisson distribution with a given density. The spatial distribution of very large drops with exponent $D(r) = 0$ is trivial: there are a few (if any) single drops randomly located. Therefore, we will focus here on large drops with exponents $0 < D(r) < 1$.

3. Simulation

The most natural way to simulate spatial distribution of drops with scaling properties satisfying Eq. (1) is to use a threshold defined by a parameter D in turbulence cascade models. It is known (e.g., Schertzer and Lovejoy 1989; Chhabra et al. 1989) that the probability of a d-dimensional cascade field φ_l at scale l to exceed a singularity of order γ is proportional to l^{d-D}, that is,

$$\text{Prob}(\varphi_l \geq l^{-\gamma}) \propto l^{d-D}. \quad (2)$$

Here D and γ are nontrivially related: namely, $D = D(\gamma)$ is the fractal dimension of the subset of φ_l with singularity strength γ. Indeed, if both parts of Eq. (2) are multiplied by the total number of boxes, $1/l^d$, then on the left side one gets the number of boxes with singularity strength between γ and $\gamma + \Delta\gamma$, while on the right side it will be l^{-D}. Assuming for simplicity $d = 1$, we get two limiting cases of $D = 0$ and $D = 1$, describing extreme events of single isolated points and a densely filled support of φ (Richtmyer 1978, p. 51), respectively. In order to simulate a set with dimension D, therefore, one can generate a one-dimensional cascade (e.g., Meneveau and Sreenivasan 1987) and then at scale l select a singularity level γ that corresponds to a given dimension D. The spatial distribution of points that are located on the intersection of the threshold $l^{-\gamma}$ (a line) and the cascade field φ_l will have the dimension D in the process of $l \to 0$.

The upper panel in Fig. 3 shows a 12-cascade p model (Meneveau and Sreenivasan 1987) with $p = 0.35$. For this simple model, there is an analytical relationship between the dimension D of the set and its singularity level γ. As an example, a threshold in the upper panel cuts a set of 79 clustered points shown in the lower part of the panel as small squares. Its dimension is estimated to be 0.3 (lower panel) and corresponds to $\gamma = 2/3$. The transition to a slope of -1 for large scales, seen in the lower panel, is due to a finite size of the interval in which points are located.

Similar to a one-dimensional cascade model (that lies on a plane), one can use two- and three-dimensional cascades that lie in three- and four-dimensional spaces, respectively. The dimension of the set with singularity strength γ can still be described by Eq. (2). In the case of a three-dimensional cascade model, an intersection of the cascades by a three-dimensional plane results in a set of points randomly distributed in space. These points will be clustered in three-dimensional space in a

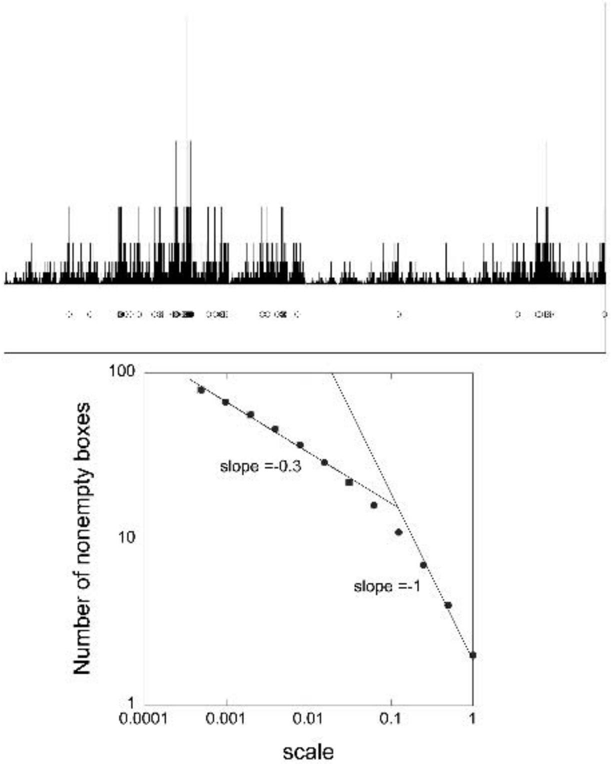

FIG. 3. (top) A simple log-binomial cascade model called the p model (Meneveau and Sreenivasan 1987). Twelve cascades with $p = 0.35$ are used. An example of a threshold with a singularity strength $\gamma = 2/3$ is shown as the horizontal dashed line. Points on the horizontal axis at which values of the cascade field exceed the threshold are depicted as small squares. There are 79 of them and they are obviously clustered. (bottom) A log–log plot of the number of nonempty boxes of scale l needed to cover these 79 points vs the scale l. The small-scale slope gives the dimension D of the points. For this simple cascade model an analytical relationship between fractal dimension D and the singularity γ can be found in Meneveau and Sreenivasan (1987). According to this relationship $\gamma = 2/3$ corresponds to $D \approx 0.3$.

way similar to the ones clustered on a line in the upper panel of Fig. 3. The resulting degree of clustering is defined by the singularity strength γ and thus by the fractal dimension D. Figure 4 illustrates this process. The upper panel shows the spatial distribution of more than 20 000 large drops as an intersection of a three-dimensional plane and a three-dimensional cascade model. It follows from scaling behavior of nonempty boxes (lower panel) that the mean number of drops N in volume V varies with V as V^D (KMLWMM). This conflicts with a fundamental assumption not only of most models, but also instrument designs, field observational strategies, and data processing; namely that the mean number of drops in volume V is proportional to V. The fractal dimension D of these drops is 0.56, which coincides with the one observed during a 2-h flight on 3 March 2000 (see Fig. 2 and its analysis shown in Fig. 1).

Finally, Fig. 5 illustrates spatial distributions of 5115 drops for two values of the fractal dimension. The gray drops are distributed uniformly. Their fractal dimension is 1, this is an implicit assumption behind most of the current radiative transfer theories in a cloudy atmosphere. By contrast, the black drops are clustered and their spatial variation follows Eq. (1) with D close to 0.55. As a result, the frequency of occurrence of black drops along a typical line is lower than the frequency of occurrence of gray drops. However, the mean number of black drops in nonempty boxes (clusters) is larger than that of gray drops.

Consider a tube with a cross section thick enough to capture the three-dimensional structure of drop spatial distribution. Since the distributions of the black (and gray) drops are isotropic by construction (i.e., there is no preferential direction), any multiple-bend tube that is long enough will show a spatial distribution statisti-

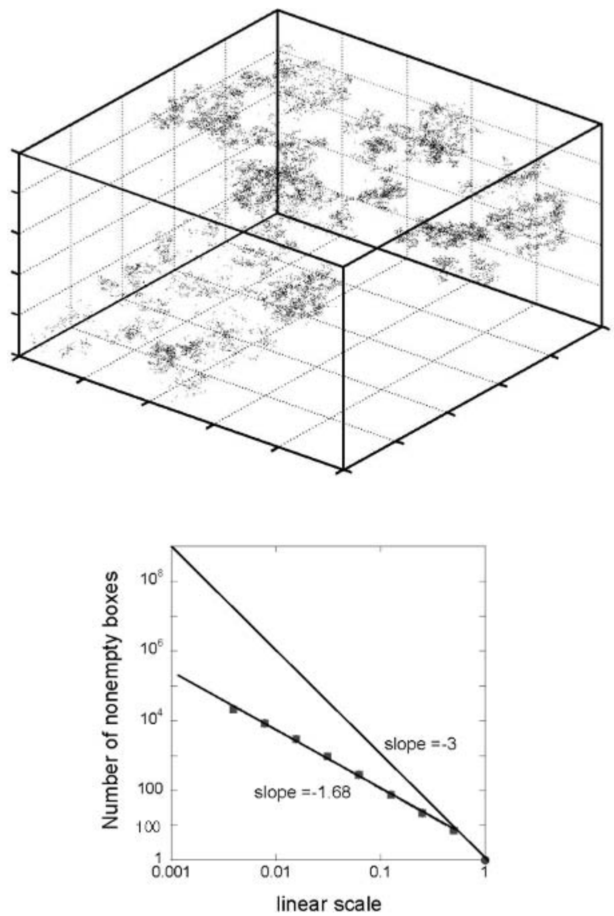

FIG. 4. A simulated "cloud" piece with more than 2×10^4 (actually 21 058) large drops. A three-dimensional 8-cascade model with the total number of pixels $2^{24} \approx 2 \times 10^8$ was used. (top) A four-dimensional cutoff at a singularity level that gives fractal dimension $D = 0.56$ has been used to simulate the spatial distribution of drops. (bottom) A straight line on a log–log plot of the number of nonempty boxes vs scale, i.e., $N(l) \propto V^{-D} = l^{-3D}$. Note that the observed variation in $N(l)$ along the flight path for drops with radii $r = 23 \pm 2$ μm exhibits similar behavior (Fig. 1).

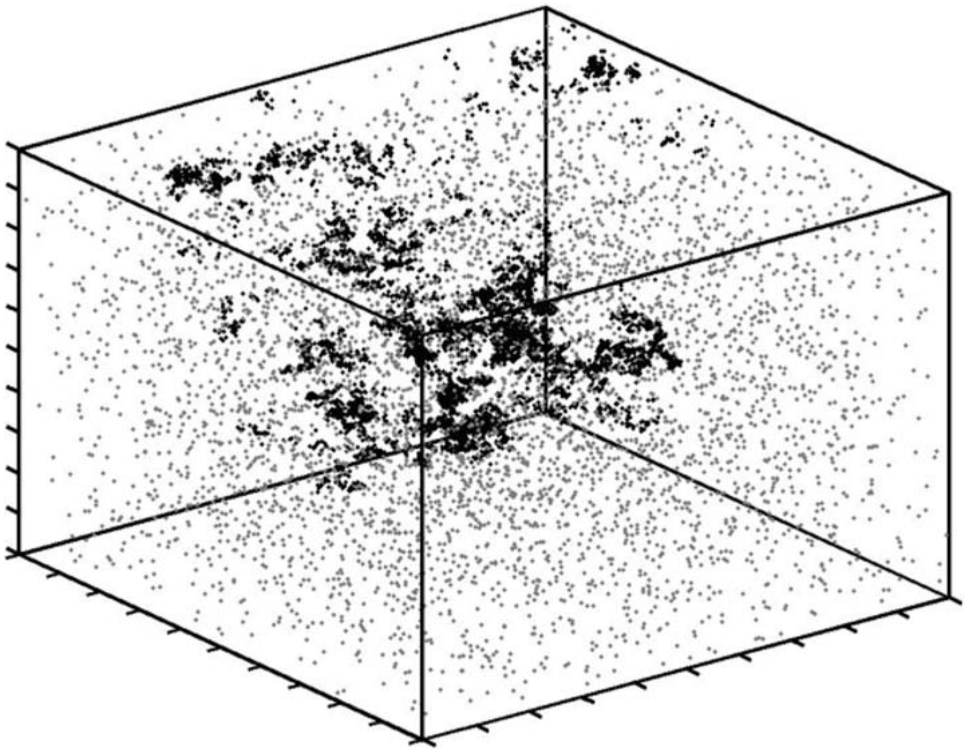

FIG. 5. An example of the space-filling properties of distributions with different fractal dimensions. Here are an equal amount of drops (total number of 5115) that are colored gray and black. The gray drops are distributed perfectly randomly throughout the space (having a fractal dimension close to 1) whereas the black particles are distributed in a such way that their fractal dimension is significantly less than 1 (between 0.5 and 0.55). The spatial distribution of black drops was simulated as an intersection of a three-dimensional plane and a three-dimensional 8-cascade model imbedded in a four-dimensional space.

cally similar to that observed during the cloud IOP (Fig. 2). By statistical similarity here we understand that the number of large drop samples follows a power law (1) with exponents D close to ½. Unfortunately, a simulated three-dimensional spatial distribution of drops (8 cascades requires about 20 million points) does not provide us with an adequate scaling range to observe a power-law behavior similar to the one in Fig. 1. Instead, we ran a one-dimensional 23-cascades model with the same fractal dimension $D = 0.56$. Figure 6 illustrates the results. In addition, a case of a perfectly random distribution of the same number of drops is also shown.

Thresholding multiplicative cascades is not the only way to simulate cloud drops whose spatial distribution follows Eq. (1). Another natural technique is to use an additive Levy flight (e.g., Mandelbrot 1982, 132–143): a sequence of jumps that are statistically independent segments whose length follows the probability distribution

$$\mathrm{Prob}(X > x) \propto x^{-\alpha}(0 < \alpha < 2); \quad (3)$$

that is, the number of jumps exceeding x is a hyperbolic distribution with parameter α. Note that for Levy flights all moments of order $k > \alpha$ diverge. The limiting case of $\alpha = 2$ corresponds to a Gaussian distribution, thus its random walk corresponds to Brownian motion where all jumps are normally distributed. The case $\alpha = 1$ is the Cauchy distribution: the behavior is dominated by one or two large "jumps." Decreasing α makes the long segments longer and the short segments shorter, thus increasing clustering. An example of this distribution for simulating rain drops can be found in Lovejoy and Mandelbrot (1985). For the relationship between a hyperbolic parameter α from (3) and a fractal dimension D defined in (1) see Mandelbrot (1982).

In general, the exponent D determines the type of the distribution. Probability theory distinguishes three classes, or types, of distributions: absolutely continuous ($D = 1$), singular ($0 < D < 1$), and discrete ($D = 0$) distributions (e.g., Richtmyer 1978, p. 260). Each of these classes contains an infinite number of distributions. A multiplicative turbulence cascade model and the additive Levy distribution used here are just two examples from the set of singular distributions. Their common feature is the exponent D; the closer D is to 0 the closer the values of the distribution function are to the discrete (clustered) set. This is a direct consequence of the Lebesque (e.g., Richtmyer 1978, p. 260) theorem

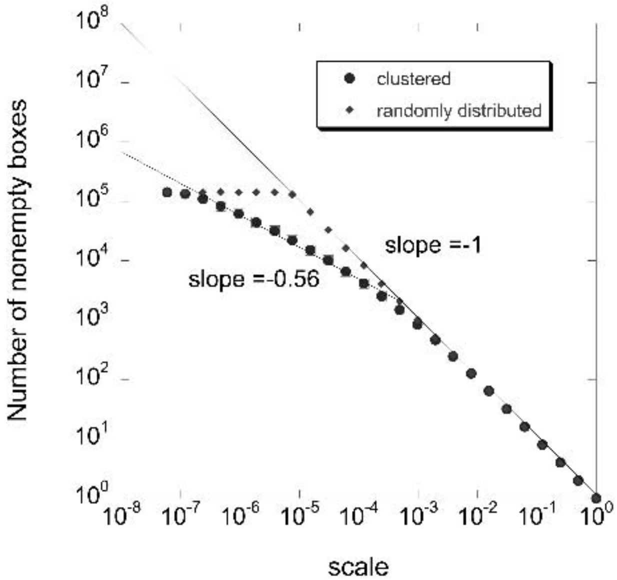

FIG. 6. Scaling behavior of nonempty boxes, N, vs scale l for a one-dimensional cascade model with 23 cascades. A power-law behavior with a scaling exponent 0.56 is well established over at least three orders of magnitude. For comparison, the N vs l curve for a perfectly random distribution of the same number of drops (142 680) is also shown. Note that at small scales ($l \to 0$), $N(l)$ is equal to the total number of drops (in this case: 142 680).

on decomposition of the distribution and the Hausdorf–Besicovitch dimension (e.g., Barnsley 1988, p. 202).

4. Discussion

The assumption that cloud drops are distributed uniformly in space and fluctuations of their number in a given volume follow Poisson statistics is built not only into most models but also into instrument design, field observational strategies, and data processing. The analysis of aircraft measurements of individual drop sizes showed that for sufficiently small volumes the mean number of drops is proportional to a drop-size-dependent nonunit power of the volume. This empirical fact leads to strong drop clustering—the lower the power is, the stronger the clustering. There are also other observational evidences of cloud-drop clustering (e.g., Shaw et al. 2002). Ignoring the clustering phenomenon can lead to underestimating the impact of large drops on cloud optical properties (KMLWMM).

Equation (1) provides a quantitative means for a size-dependent description of clustering and spatial distribution of drops. It states that the intensity of clustering is measured by the power-law exponent $0 < D < 1$. The smaller D is, the larger the degree of clustering. When $D = 1$, cloud drops are not clustered and the number of drops in a volume is proportional to the volume. Equation (1) also allows us to develop a scale-invariant model of the spatial distribution of large drops that has the same drop-size-dependent exponents as the ones observed. Below we discuss briefly the possible use of Eq. (1) and cloud-drop models in cloud physics and radiation.

The spatial distribution of cloud drops, especially large drops, is not yet fully understood and remains controversial (see Pinsky and Khain 2001, and references therein). Small-scale spatial correlation and clustering are important in cloud-drop growth rate and can help explain some of the fundamental problems in cloud physics. For example, Twomey (1976) suggested clustering ("pockets of high liquid water") for explaining observed warm rain. Once a correct theory which predicts the observed power-law statistics is in hand, the strong clustering of larger drops falls out naturally from the statistics; no deus ex machina need be invoked to explain the clustering. Recently, McGraw and Liu (2003) developed a new model for cloud drizzle formation that quantitatively explains how cloud turbulence enhances the growth of cloud droplets by both condensation and collection. In particular, they showed that once drops reach a critical radius of about 20 μm they can grow much faster through collection, transforming cloud drops to drizzle size. Classical condensation theory was unable to explain the production of these drops because of their slow growth rate.

It is most likely that atmospheric turbulence significantly enhances drop clustering (Kostinski and Shaw 2001; Shaw 2003), especially for large drops (Pinsky and Khain 1997). The scale exponents $D(r)$ depend on the small-scale variability of clouds determined by thermodynamic and fluid-mechanical interactions between droplets and the surrounding air. However, the exponents have nothing to do with cloud macroscale structure. In other words, two clouds could "look" alike dynamically but have different small-scale turbulence and different degrees of clustering, or they might look different dynamically but have similar small-scale turbulence and thus the same degree of clustering.

To see the consequences of scaling behavior (1) on cloud radiative properties, let us assume for simplicity that a cloud consists of only two types of drops: small drops (subscript S) with $D_S = 1$ and large drops (subscript L) with $0 < D_L < 1$. The mean number, $n(r,V)$, of drops with radius r in a volume V is

$$n(r,V) = n_S(V)\delta(r - r_S) + n_L(V)\delta(r - r_L), \qquad (4)$$

where

$$n_S(V) = \rho_S V^{D_S} = \rho_S V \qquad (5a)$$

and

$$n_L(V) = \rho_L V^{D_L} \qquad (5b)$$

are the mean number of small and large drops in volume V, respectively. Here ρ_S and ρ_L are volume-independent generalized drop concentrations [in number per cluster $(cm^3)^D$; see Wiscombe et al. 2003;

KMLWMM], and δ is the Dirac δ function. Substituting Eqs. (4)–(5) into the definition of droplet effective radius r_e (e.g., Hansen and Travis 1974) we get

$$r_e(V) = \frac{\rho_S V r_S^3 + \rho_L V^{D_L} r_L^3}{\rho_S V r_S^2 + \rho_L V^{D_L} r_L^2}. \quad (6)$$

It follows from Eq. (6) that for small scales, $r_e(V) \to r_L$ as $V \to 0$, while for large scales $r_e(V) \to r_S$ as $V \to \infty$. If one assumes $D_L = 1$, as does the conventional technique, we get $r_e(V) \equiv r_e$ which is typically much closer to r_S than to r_L since the concentration of large drops is negligible compared to small ones ($\rho_L << \rho_S$). In other words, the conventional technique systematically underestimates the effect of large drops at small scales. The effective radius of drops in a cloud estimated under the assumption of uniformly distributed drops (e.g., gray dots in Fig. 5) is almost always smaller than the one for a cloud with clustered drops (e.g., black dots in Fig. 5). This suggests a partial explanation for the fact that r_e retrieved from satellites is usually larger than the one measured in situ [e.g., Dong et al. (2002) for the March 2000 ARM Cloud IOP; also Figs. 1 and 2].

5. Conclusions

This paper shows a new empirical way of specifying the spatial clustering of droplets in a cloud, based on a new kind of analysis of FSSP data grounded in (multi)-fractal theory. We have discovered that this clustering is a function of drop size r, and that it can be encapsulated into a single function $D(r)$ whose physical meaning is that of a dimension. This function generalizes and makes more quantitative several decades of work on clustering going back at least to Twomey (1976), if not further. Using only $D(r)$, it is possible to create realizations of the location of every drop in a cloud as a function of its size, and all such realizations naturally exhibit clustering of drops bigger than about 14 μm without introducing such clustering as an arbitrarily imposed constraint.

Our empirical theory can be regarded as being intermediate between the simpler empirical clustering theories of yore, which have advanced little in 30 years and were more descriptive than quantitative, and a full solution of the turbulence problem for a nearly colloidal suspension, which presumably could derive clustering from first principles. While it would be nice to have the latter full solution, decades of work on it have not led to much progress, so in the meantime we suggest that much more rapid progress can be made using our intermediate theory, which is solidly grounded in observation, relatively simple, and sufficient to model the locations of droplets within any cloud.

There are many applications for this new formulation, not the least of which is the warm rain problem, which has frustrated cloud physicists for 40 years. Twomey recognized that the warm rain problem would be insoluble without clustering, but did not have the highly time-resolved observations needed to make much progress on clustering, nor the (multi)fractal analysis tools that were developed a decade later and only first applied to high-time-resolution FSSP data by us.

Acknowledgments. This research was supported by the Department of Energy (under Grant DE-AI02-95ER61961 to NASA's GSFC) as part of the Atmospheric Radiation Measurement (ARM) program and by the NASA Radiation Science Program (under Grant NNG04GF15G to Boston University). M. Larsen was supported by the Department of Defense's National Defense Science and Engineering Graduate Fellowship Program. He also thanks the Goddard Summer Student Program for its support of this research and his stay at NASA GSFC for the summer of 2003.

REFERENCES

Baker, B., 1992: Turbulent entrainment and mixing in clouds: A new observational approach. *J. Atmos. Sci.,* **49,** 387–404.

Barnsley, M., 1988: *Fractals Everywhere.* Academic Press, 402 pp.

Brenguier, J.-L., T. Bourianne, A. Coelho, J. Isbert, R. Peytavi, D. Trevarin, and P. Wechsler, 1998: Improvements of droplet size distribution measurements with the Fast-FSSP. *J. Atmos. Oceanic Technol.,* **15,** 1077–1090.

Chhabra, A. B., C. Meneveau, R. V. Jensen, and K. R. Sreenivasan, 1989: Direct determination of the $f(\alpha)$ singularity spectrum and its application to fully developed turbulence. *Phys. Rev.,* **40A,** 5284–5294.

Davis, A., A. Marshak, H. Gerber, and W. Wiscombe, 1999: Horizontal structure of marine boundary-layer clouds from cm- to km-scales. *J. Geophys. Res.,* **104,** 6123–6144.

Dong, X., P. Minnis, G. G. Mace, W. L. Smith Jr., M. Poellot, R. T. Marchan, and A. D. Rapp, 2002: Comparison of stratus cloud properties deduced from surface, GOES, and aircraft data during the March 2000 ARM Cloud IOP. *J. Atmos. Sci.,* **59,** 3265–3284.

Hansen, J. E., and L. D. Travis, 1974: Light scattering in planetary atmospheres. *Space Sci. Rev.,* **16,** 527–609.

Hobbs, P. V., and A. L. Rangno, 1985: Ice particles concentrations in clouds. *J. Atmos. Sci.,* **42,** 2523–2549.

Jameson, A. R., A. B. Kostinski, and R. A. Black, 1998: The texture of clouds. *J. Geophys. Res.,* **103,** 6211–6220.

Knyazikhin, Y., A. Marshak, W. J. Wiscombe, J. Martonchik, and R. B. Myneni, 2002: A missing solution to the transport equation and its effect on estimation of cloud absorptive properties. *J. Atmos. Sci.,* **59,** 3572–3585.

——, ——, M. L. Larsen, W. J. Wiscombe, J. Martonchik, and R. B. Myneni, 2005: Small-scale drop size variability: Impact on estimation of cloud optical properties. *J. Atmos. Sci.,* in press.

Kostinski, A. B., and A. R. Jameson, 1997: Fluctuation properties of precipitation. Part I: On deviations of single-size drop counts from the Poisson distribution. *J. Atmos. Sci.,* **54,** 2174–2186.

——, and ——, 2000: On the spatial distribution of cloud particles. *J. Atmos. Sci.,* **57,** 901–915.

——, and R. A. Shaw, 2001: Scale-dependent droplet clustering in turbulent clouds. *J. Fluid Mech.,* **434,** 389–398.

Liu, Y., and J. Hallett, 1998: On size distributions of cloud droplets growing by condensation: A new conceptual model. *J. Atmos. Sci.,* **55,** 527–536.

——, P. H. Daum, and J. Hallett, 2002: A generalized systems theory for the effect of varying fluctuations on cloud droplet size distribution. *J. Atmos. Sci.,* **59,** 2279–2290.

Lovejoy, S., and B. B. Mandelbrot, 1985: Fractal properties of rain and a fractal model. *Tellus,* **37A,** 209–232.

Mandelbrot, B., 1982: *The Fractal Geometry of Nature.* W. H. Freeman, 460 pp.

McGraw, R., and Y. Liu, 2003: Kinetic potential and barrier crossing: A model for warm cloud drizzle formation. *Phys. Rev. Lett.,* **90,** 18501, doi:10.1103/PhysRevLett.90.018501.

Meneveau, C., and K. R. Sreenivasan, 1987: Simple multifractal cascade model for fully developed turbulence. *Phys. Rev. Lett.,* **59,** 1424–1427.

Pawlowska, H., and J.-L. Brenguier, 1997: Optimal nonlinear estimation for cloud particle measurements. *J. Atmos. Oceanic Technol.,* **14,** 88–104.

Pinsky, M., and A. P. Khain, 1997: Formation of inhomogeneity in drop concentration induced by the inertia of drops falling in a turbulent flow, and the influence of the inhomogeneity on the drop-spectrum broadening. *Quart. J. Roy. Meteor. Soc.,* **123,** 165–186.

——, and ——, 2001: Fine structure of cloud droplet concentration as seen from the Fast-FSSP measurements. Part I: Method of analysis and preliminary results. *J. Appl. Meteor.,* **40,** 1515–1537.

——, and ——, 2003: Fine structure of cloud droplet concentration as seen from the Fast-FSSP measurements. Part II: Results of in situ observations. *J. Appl. Meteor.,* **42,** 65–73.

Richtmyer, R. D., 1978: *Principles of Advanced Mathematical Physics.* Vol. 1, Springer-Verlag, 422 pp.

Schertzer, D., and S. Lovejoy, 1989: Generalized scale invariance and multiplicative processes in the atmosphere. *Pure Appl. Geophys.,* **130,** 57–81.

Shaw, R. A., 2003: Particle-turbulence interactions in atmospheric clouds. *Annu. Rev. Fluid Mech.,* **35,** 183–227.

——, A. B. Kostinski, and M. L. Larsen, 2002: Towards quantifying droplet clustering in clouds. *Quart. J. Roy. Meteor. Soc.,* **128,** 1043–1057.

Twomey, S., 1976: The effects of fluctuations in liquid water content on the evolution of large drops by coalescence. *J. Atmos. Sci.,* **33,** 720–723.

Wiscombe, W., Y. Knyazikhin, and A. Marshak, 2003: A new look into the treatment of small-scale drop variability in radiative transfer. *Proc. 13th Atmospheric Radiation Measurement (ARM) Science Team Meeting,* Broomfield, CO, U.S. Dept. of Energy. [Available online at http://www.arm.gov/publications/proceedings/conf13/extended_abs/wiscombe-wj.pdf.]

Young, K. C., 1993: *Microphysical Processes in Clouds.* Oxford University Press, 427 pp.